Complex Analysis Questions

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Based on Lectures in Spring 2008, last updated May 13, 2008

0 Introduction

This file includes the questions from Complex Analysis, only chapters 2 and later are included as I've already do the questions in Chapter 1. These questions are NOT official, and there is no guarantee that they are correct. They are released purely in the hope that they will be useful. Corrections are welcome and you will be credited (or not if you tell me) in this document. Send me an email to matthew@matthewhutton.com with any corrections you have.

This file is only provided as a courtesy, to be honest I am not great at copying stuff down, so if there are mistakes in the questions they are probably due to inaccurate copying, which I am not responsible for. If you do find your notes are different drop me an email.

Note that some questions are numbered differently from the lectures, these changes are pointed out in the footnotes.

0.1 Changelog

17/01/08 - Initial Version, questions up to end of week 2.

20/01/08 - Formatting correction to Question 2.10.

22/01/08 - Questions up to the 22/01/08 added, minor formatting correction to question 3.1

23/01/08 - Questions 3.5 and 4.4 and 4.2 corrected.

28/01/08 - Questions up to 28/01/08 added.

29/01/08 - Pagebreak for Chapter 7 questions added, questions up to 29/01/08 added. Fixed issue with question 3.7 so it now works.

30/01/08 - Corrected mistake in Question 3.8.3

31/01/08 - Questions up to 31/01/08

06/02/08 - ‘uestions up to 05/02/08

10/02/08 - Corrected mistake in Question 6.2

12/02/08 - Questions up to 12/02/08

16/02/08 - Changed Support to Suppose in Question 6.4 so the sentence makes sense, improved wording in Question 6.5.

19/02/08 - Questions up to 19/02/08. Question 8.6 is corrected so it makes sense. Added part 5 to Question 9.1

21/02/08 - Questions up to 21/02/08.

25/02/08 - Questions up to 25/02/08

26/02/08 - Questions up to 26/02/08, Question 10.7 corrected for clarity.

3/03/08 - Questions up to 3/03/08

4/03/08 - Questions up to 4/03/08, Slash removed from end of question 10.3. Question 3.9 corrected

6/03/08 - Questions up to 6/03/08, How spelt correctly in question 11.12, footnotes in section 11 clarified.

10/03/08 - All questions now included, questions up to 10/03/08 added, Hints separated.

11/03/08 - Missed Questions completely removed (but the numbering jumps one).

17/03/08 - Minor Change for clarity to the last numbering footnote on page 14.

4/04/08 - Question 4.5 includes all the parts it should have, not just the last one.

12/05/08 - Minor formatting correction to question 9.1.5. question 8.2 clarified.

13/05/08 - Question 12.4 and 11.3 corrected.

1 The Complex Number System

Question 1.1. Show that $\mathbb{C}$ is a field, with neutral elements $(0, 0)$ and $(1, 0)$ respectively.

Question 1.2. Show the following:

1. $\operatorname{Re} z = \frac{z+\bar{z}}{2}$
2. $\operatorname{Im} z = \frac{z-\bar{z}}{2}$
3. $\overline{z+w} = \bar{z} + \bar{w}$
4. $\overline{zw} = \bar{z}\bar{w}$
5. $|zw| = |z||w|$

Question 1.3. Find the real and imaginary parts of:

1. $1/z$
2. z^3
3. $\frac{3+5i}{1+7i}$
4. i^n $(2 \leq n \leq 8)$

Question 1.4. Find the absolute and conjugate of:

1. $3 + i$
2. -6
3. $\frac{3-i}{\sqrt{2+3i}}$
4. i^9

Question 1.5. Show that $z\ in \mathbb{R} \leftrightarrow z = \bar{z}$

Question 1.6. Show the following:

1. $|z + w|^2 = |z|^2 + 2\mathrm{Re}\,(z\bar{w}) + |w|^2$
2. $|z - w|^2 = |z|^2 - 2\mathrm{Re}\,(z\bar{w}) + |w|^2$
3. $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$

Question 1.7. Prove by induction

$$|z_1 z_2 \cdots z_n| = |z_1||z_2| \dots |z_n|$$

Question 1.8. Let $R(z)$ be a rational function of z. Show that if all the coefficients of $R(z)$ are real then $\overline{R(z)} = R(\bar{z})$

Question 1.9. Prove Cauchy's inequality:

$$\left|\sum_{i=1}^{n} z_i w_i\right| \leq \left(\sum_{i=1}^{n} |z_i|\right)\left(\sum_{i=1}^{n} |w_i|\right)$$

Question 1.11. Prove de Moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Question 1.12. Find the square root of z. i.e. find w such that $w^2 = z$.

Question 1.13. Prove the quadratic equation. i.e. solve $az^2 + bz + c = 0$ for $z \in \mathbb{C}$, constants $a, b, c \in \mathbb{C},\ a \neq 0$.

Question 1.14. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ $\gamma \neq 0$, $\alpha\delta \neq \beta\gamma$ Define:

$$\phi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

Show that $\phi : \mathbb{C}\backslash \left\{\frac{-\delta}{\gamma}\right\} \to \mathbb{C}\backslash \left\{\frac{\alpha}{\gamma}\right\}$ is a bijection which maps circles to circles.

Question 1.15. What about circles through $-\delta/\gamma$?

Question 1.16. Show that the system of matrices of the form:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

With $a, b \in \mathbb{R}$ and matrix addition and multiplication form a field which is isomorphic to $\mathbb{C}$.

2 Topology of the complex plane

Question 2.1. Determine which of the following sets is open, closed or neither:

1. $1 < |z| < 3$
2. $\operatorname{Re}(z) \geq 0$
3. $\operatorname{Im}(z) \geq 0$
4. $1 < |z| < 2$
5. $\operatorname{Re}(z^2) > 0$
6. $0 < \arg(z) < \frac{\pi}{2}$

Question 2.2. Prove the following

$$\lim_{z \to z_0} f(z) = l \Leftrightarrow \lim_{z \to 0} \operatorname{Re} f(z) = \operatorname{Re} l \text{ and } \lim_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} l$$

Question 2.3. Find the limits if they exist as $z \to 0$ of

1. $\frac{|z|}{z}$
2. $\frac{|z|^2}{z}$
3. $\frac{z^2}{|z|^2}$
4. $\frac{z - \operatorname{Re} z}{\operatorname{Im} z}$

Question 2.4. Prove proposition 2.4 in the lecture notes.

Question 2.5. Prove proposition 2.6 in the lecture notes.

Question 2.6. Show that $|z|$, $\operatorname{Re} z$, $\operatorname{Im} z$, $z^2 + |z|$, $|z|^2$ are continuous functions.

Question 2.7. Give an explicit function for the curve going along the line at 0 degrees from a point at radius ϵ to a point at radius R, with an arc at radius R to the line at angle 45 degrees, then along the line at 45 degrees from radius R to radius ϵ, then finally going down an arc at radius ϵ.

Question 2.8. Prove proposition 2.11 in the lecture notes.

Question 2.9. Let $S \subseteq \mathbb{C}$ be (path) connected and $f : S \to \mathbb{C}$ continuous. Then $f(S)$ is connected.

Question 2.10. Identify the boundary of:

1. $S = \{z \in \mathbb{C}, |z| \neq 1\}$
2. $S = \{z \in \mathbb{C} \big| 1 \leq |z| \leq 2,\ 0 \leq \operatorname{Im} z \leq \operatorname{Re} z\}$

In the cases where ∂S can be described as the image of a path. Specify a function giving the path.

3 Power Series

Question 3.1.

1. A series $\sum z_r$ is convergent $\Leftrightarrow \sum \operatorname{Re} z_r$ and $\sum \operatorname{Im} z$ are convergent, in which case $\sum z_r = \sum \operatorname{Re} z_r + i \sum \operatorname{Im} z_r$.

2. $\sum z_r$ and $\sum w_r$ convergent implies $\sum (z_r + w_r)$ convergent to $\sum z_r + \sum w_r$ and $\sum c z_r = c \sum z_r$ $\forall c \in \mathbb{C}$.

Question 3.2. $\sum z_r$ absolutely convergent $\Rightarrow \sum z_r$ convergent.

Question 3.4.

1. Show that if $\sum a_n z^n$ converges for $z = z_1 \neq 0$ then it converges absolutely $\forall z \in \mathbb{C}$ s.t. $|z| < |z_1|$

2. If $\sum a_n z^n$ diverges at $z = z_2$ then it diverges $\forall z$ with $|z| > |z_2|$

Question 3.5. If $\sum a_n$ and $\sum b_n$ are absolutely convergent and we set $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$ then $(\sum a_n)(\sum b_n) = \sum c_n$

Question 3.6. Show that $\exp(z+w) = \exp(z)\exp(w)$ and deduce that $\cos(z+w) = \cos z \cos w - \sin z \sin w$

Question 3.7. Prove that for $z \neq 1$ that:

$$\sum_{r=1}^{n} \frac{z^r}{r} = \frac{z}{1-z} \left(\sum_{r=1}^{n-1} \frac{1}{r(r+1)} - \sum_{r=1}^{n-1} \frac{z^r}{r(r+1)} + \frac{1-z^n}{n} \right)$$

Question 3.8. Show that:

1. $\sum \frac{z^r}{r}$ and $\sum \frac{z^r}{r(r+1)}$ have $R = 1$

2. $\sum \frac{z^r}{r(r+1)}$ converges everywhere on $|z| = 1$.

3. $\sum \frac{z^r}{r}$ converges everywhere on $|z| = 1$ except at $z = 1$.

Question 3.9. Show that $\sum z^{n!}$ has $R = 1$ but diverges for infinitely many z on $|z| = 1$.

4 Differentiation

Question 4.1. Prove the reverse of proposition 4.5

Question 4.2. Prove that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

Question 4.3. Use the following equation to prove $e^{z+w} = e^z e^w$ using the formula for multiplying two series.

$$\frac{d}{dt}e^z = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

Question 4.4. Define $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$, $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$, tan, cot, cosec and sec, are defined as at A Level in the real case. Sinh and Cosh are also defined as the following: $\sinh(z) = \frac{e^z - e^{-z}}{2}$, $\cosh(z) = \frac{e^z+e^{-z}}{2}$.

Find the derivatives of these functions.

Question 4.5. Show the following:

1. $\int_{\gamma+\sigma} f = \int_\gamma f + \int_\sigma f$
2. $\int_{-\gamma} f = -\int_\gamma f$
3. $\int_\gamma (f+g) = \int_\gamma f + \int_\gamma g$
4. $\int cg = c\int g, \ \forall c \in \mathbb{C}$

6 Integration

Question 6.1. Prove the following result:

If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges for $|z - z_0| < R$, then $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$ converges for $|z - z_0| < R$

Question 6.2. Show that for $f(z) = |z|$ the integral $\int_\gamma f$ can depend on γ (not just on the end points)

Question 6.3. For $\gamma = e^{it}$ $(0 \geq t \geq \pi)$ evaluate $\int_\gamma f$ for:

1. $f(z) = \frac{1}{z^2}$
2. $f(z) = \cos z$
3. $f(z) = \sinh z$

Question 6.4. Suppose f, g have continuous derivatives in a domain D and γ is a contour in D from z to w. Show that $\int_\gamma fg' = f(w)g(w) - f(z)g(z) - \int_\gamma f'g$

Question 6.5. Let $C_r(t) = z_0 + re^{it}$ $(0 \leq t \leq 2\pi)$ $z_0 \in D$ (domain) and that $f : D \to \mathbb{C}$ is continuous. Use the following result:

If $f : D \to \mathbb{C}$ is continuous and γ is a contour in D of length L and $|f(z)| \leq M$ on $\gamma : [a, b]$ then $\left|\int_\gamma f\right| \leq ML$.

To prove the following:

1. $\lim_{r\to 0} \int_{C_r} f(z) = 0$
2. $\lim_{r\to 0} \int_{C_r} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

7 The Winding Number

Question 7.1. Find a continuous choice of the argument $\theta : [0, 6\pi] \to \mathbb{R}$ for the path $\gamma(t) = e^{-it}$ $(0 \leq t \leq 6\pi)$ and compute $w(\gamma, 0)$.

Question 7.2. Show that $w(\gamma_1 + \gamma_2, 0) = w(\gamma_1, 0) + w(\gamma_2, 0)$[1]

Question 7.3. Let $S = \mathbb{C} \backslash \gamma([a, b])$ where $\gamma : [a, b] \to \mathbb{C}$ is a closed path, show that $z \mapsto w(\gamma, z)$ is constant on each connected component of S.

Hint (Question 7.3)**.** You know $w(\gamma, z_0) \in \mathbb{Z}$. If it is integer valued it can only be continuous if it is constant. So you just have to prove it is continuous on each connected component of S.

[1] I believe from reading theorem 7.2 in Stewart and Tall that this requires the end of γ_1 to be the beginning of γ_2. I will check with the lecturer on Monday.

8 Cauchy's Theorem

Question 8.1. Suppose γ_1 and γ_2 are contours in D and that they have the same winding number around all points outside D. Show that:

$$\int_{\gamma_1} f = \int_{\gamma_2} f \quad \forall \text{ differentiable } f$$

(Use Cauchy's Theorem)

Question 8.2. Given a domain D and differentiable function $f` : D \to \mathbb{C}$, show that $\int_\gamma f = 0$ for all closed contours γ in D if and only if D is simply connected (i.e. no holes).

Question 8.3. Let $D = \mathbb{C}\backslash\{0\}$ For $z_0 \in D$ specify a local anti-derivative i.e. some neighbourhood of z_0 for each of the following.

1. $f(z) = \frac{1}{z}$
2. $f(z) = \frac{1}{z^2}$
3. $f(z) = \frac{\cos z}{z}$

Question 8.4. Let $D = \{z \in \mathbb{C} \big| z \neq \pm i\}$ Find all possible values of $\int_\gamma \frac{1}{z^2+1}$ where γ is a closed contour in D.

Question 8.5. Let $\gamma_1 = S_1 + L - S_2 - L$, $\gamma_2 = S_1 + L + S_2 - L$ where $S_1(t) = e^{it}$ $(0 \leq t \leq 2\pi)$, $S_2(t) = 2e^{it}$ $(0 \leq t \leq 2\pi)$ and $L = [1, 2]$. Describe the inside and outside of γ_1 and γ_2. Let $f(z) = \frac{\cos z}{z}$ and compute $\int_{\gamma_1} f$ and $\int_{\gamma_2} f$.

Hint (Question 8.5)**.** Using power series for $\cos z$, write $f(z) = \frac{1}{z} + g(z)$ where $g(z)$ is another function.

Question 8.6. Show that there is a continuous version of the square root on $C_\alpha = \mathbb{C}\backslash\{re^{i\alpha} \big| r \geq 0\}$. A square root is a function $h(z)$ s.t. $\big(h(z)\big)^2 = z$.

9 Taylor Series

Question 9.1. Find the taylor series at zero, and the radius of convergence for the following:

1. $f(z) = \log(1+z)$
2. $f(z) = \frac{1}{1+z}$
3. $f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$
4. $f(z) = \exp\left(\frac{z}{1-z}\right)$
5. $F(z) = \int_0^z e^{w^2} dw$ [2]

Question 9.2. Let $\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} F_n z^n$. Show that $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Expanding $(1-z-z^2)^{-1}$ in partial fractions show that

$$F_n = \frac{1}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Question 9.3. Use Cauchy's integral formula to show that if f is differentiable on a domain D and $N_R(z_0) \subseteq D$ then for $0 < r < R$

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{it}\right) dt = f(z_0) \quad \text{“Mean value property”}$$

Question 9.4. Show that if f is analytic throughout $\mathbb{C}$ and $|f(z)| \leq |z^n|$ $\forall z \in \mathbb{C}$ where n is a positive integer, then f is a polynomial of degree at most n.

Question 9.5. Show that if f is analytic throughout $\mathbb{C}$ then it is not possible to have $|f^{(n)}(0)| > n!n^n$ for all n.

[2] Note this part was added in the seminar on 19th February 2008, and was never given out in a lecture.

10 Laurent Series

Question 10.1. Show that the Laurent Series of the following theorem is unique.

If f is differentiable in the annulus $R_1 < |z - z_0| < R_2$ where $0 \leq R_1 < R_2 \leq +\infty$ then $f(z_0 + h) = \sum_{n=0}^{\infty} a_n h^n + \sum_{n=1}^{\infty} b_h^{-n}$ where $\sum a_n h^n$ converges for $|h| < R_2$ and $\sum_{n=1}^{\infty} b_n h^{-n}$ converges for $|h| > R_1$. (so that both series converge inside the annulus). Furthermore, if $R_1 < r < R_2$ and $C_r(t) = z_0 + re^{it}$ for $0 \leq t < 2\pi$ then $a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}$, $b_n = \frac{1}{2\pi i} \int_{C_r} f(z)(z - z_0)^{n-1} dz$

i.e. if $f(z_0 + h) = \sum_{n=-\infty}^{\infty} c_n h^n = \sum_{n=-\infty}^{\infty} d_n h^n$ for $R_1 < |z - z_0| < R_2$ then $c_n = d_n \; \forall n \in \mathbb{Z}$.

Question 10.2. Find Laurent series for the following around $z = 0$:

1. $(z - a)^k \; (a \in \mathbb{C}, \; k \in \mathbb{N})$
2. $\frac{1}{z(1-z)}$
3. $e^{z+\frac{1}{z}}$
4. $z^{-4} \sin z$

Question 10.3. Show the following:

1. $e^{\frac{1}{z}}$ for $0 < |z| < r$ takes every value in $\mathbb{C}$ with one exception.
2. $\sin\left(\frac{1}{z}\right)$ $0 < |z| < r$ takes every value in $\mathbb{C}$.

Question 10.4. Show a function f is meromorphic on $\bar{\mathbb{C}}$ if f it is a rational function $f(z) = \frac{p(z)}{q(z)}$ where p and q are polynomials.

Question 10.5. Find the poles and zeros of $\tan z$ show that $\tan z$ is meromorphic on $\mathbb{C}$ but is not rational.

Question 10.6. If f is analytic on $\bar{C}$ and doesn't have any zeros $\{f(z) \neq 0 \; \forall z \in \bar{\mathbb{C}}\}$ then f is constant.

Question 10.7 (Challenge Question).
If $f : \mathbb{R} \to \mathbb{R}$ s.t. $|f^{(n)}(x)| \leq 1 \; \forall n \geq 0$ and $f'(0) = 1$ then $f(x) = \sin x$.[3]

[3]The first person to complete this will win a prize.

11 Residues

Remark. The numbering in this sheet is different from the lectures, check the footnotes for details.

Question 11.1. Compute:

$$\int_0^{2\pi} (\cos^a t + \sin^b t)dt$$

For the following:

1. $a = 1$, $b = 2$
2. $a = 4$, $b = 4$

Question 11.2. Evaluate:

$$\int_0^{\pi} \frac{1}{2 + \cos t} dt$$

Question 11.3. Evaluate:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx$$

Question 11.4. Evaluate:

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + a^2)^2} dx$$

Question 11.5. Let $a \notin \mathbb{Z}$ then prove the following:

1. $\frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}$
2. $\pi \cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$

Question 11.6. If[4] f is an entire function with $|f(z)| \leq e^{|z|}$ for all $|z|$ sufficiently large, then for $z, t \in \mathbb{C}$ with $\frac{z-t}{\pi} \notin \mathbb{Z}$

$$\frac{f'(z)\sin(z+t) - f(z)\cos(z+t)}{\sin^2(z+t)} = -\sum_{n=-\infty}^{\infty} \frac{f(\pi n - t)}{(z - \pi n + t)^2}$$

Question 11.7. How[5] many zeros of $z^4 + 4z^3 + 6z^2 - 4z + 3$ lie inside the disc $|1 - z| < 1$

Question 11.8. If[6] $|a| > e$ show that $e^z = az^n$ has n roots in the unit disc.

Question 11.9. Determine[7] explicitly the largest disc centred at the origin on which the function $f(z) = z^2 + z$ is one to one.

Question 11.10. How many roots of $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ lie in the unit ($|z| < 1$) disc?

Question 11.11. How many roots of $z^4 - 6z + 3 = 0$ lie in the annulus $1 < |z| < 2$?

Question 11.12 (Challenging)**.** How many roots of $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in $Re(z) > 0$.

[4] In the lectures this was given as the first question 11.5 part iii, but it seems like a different question so I've split it off.
[5] In the lectures this was given as the second question 11.5.
[6] In the lectures this was given as question 11.6.
[7] This and subsequent questions in section 11 are numbered one higher than given in the lectures

12 Conformal Mappings

Remark. The numbering in this sheet is different from the lectures, check the footnotes for details.

Remark. Note that $\mathbb{H}$ is the postive complex half plane, i.e. $\{z \in \mathbb{C} \big| Im(z) > 0\}$.

Question 12.1. Show that every mobius transformation can be written as a composition of translations ($z \mapsto z + t, t \in \mathbb{C}$), inversions $z \mapsto 1/z$, dilations ($z \mapsto az, a \in \mathbb{R}^+$) and rotations ($z \mapsto cz, |c| = 1$).

Question 12.2. Find[8] the function $f(z)$ for arbitrary points $z_2,\ z_3,\ z_4 \in \mathbb{C}$ that is a unique mobius transformation f s.t. $f(z_2) = 1$, $f(z_3) = 0$ and $f(z_4) = \infty$.

Question 12.3. Show[9] that for any mobius transform g and $z_1,\ z_2,\ z_3,\ z_4$ distinct that:

$$\big(g(z_1),\ g(z_2),\ g(z_3),\ g(z_4)\big) = (z_1, z_2, z_3, z_4)$$

Question 12.4. Find[10] the unique mobius transform f which maps $\{-1, 1\}$ to $\{0, 1\}$ and i to ∞.

Question 12.5. For[11] the function f defined in question 12.4. Show that $f(\{z \in \mathbb{C} \big| |z| < 1\}) = \mathbb{H}$

Question 12.6. Let[12] z_2, z_3 and z_4 be distinct and also let w_2, w_3 and w_4 be distinct. Show that there is a unique mobius transformation which maps z_2 to w_2, z_3 to w_3 and z_4 to w_4.

Question 12.7. Show[13] the following:

$$\text{Aut}(\mathbb{H}) = \left\{ f(z) = \frac{az + b}{cz + d},\ ab - bc = 1,\ a, b, c, d \in \mathbb{R} \right\}$$

Hint (Question 12.7)**.** Find a conformal one-to-one map from the disc to $\mathbb{H}$ and apply the theorem *If $f \in \text{Aut}(D)$ with $f(a) = 0$ then $f = c\phi_a$ for some c with $|c| = 1$*

13 Harmonic Functions

Question 13.1. If u is harmonic on D' then for $z \in D$ $u(z) = \frac{1}{2\pi} \int_0^{2\pi} u\left(e^{it}\right) P_z(t) dt$ where $P_z(t) = \text{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)$

Hint (Question 13.1)**.** Use Cauchy's Integral Formula

[8] In the lectures this was given as question 12.1a)

[9] In the lectures this was given as question 12.2)

[10] In the lectures this was given as question 12.2a)

[11] In the lectures this was given as question 12.3)

[12] In the lectures this was given as question 12.4

[13] In the lectures this question was unlabelled