Algebra II
Classification of Groups

Ian Rogers

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Abstract

This document lists the propositions relating to classification of groups up to order 11; then summarises these groups and their classifications in a table.

1 Propositions and proofs

Proposition 1. *Let G have order p, p prime. Then $G \cong C_p$.*

Proof. Let $1 \neq g \in G$. Then $|g| > 1$, $|g| \big| p$ so $|g| = p$. Hence $G = \langle g \rangle$. □

Proposition 2. *Let $|G| = 4$. Then $G \cong C_4$ or K_4.*

Proof. $C_4 \ncong K_4$ since C_4 has an element of order 4, K_4 does not.

Non-identity elements of G have order 2 or 4 by Lagrange.

- If $\exists\, a$ s.t. $|a| = 4$ then $G = \langle a \rangle = C_4$.
- If no such element all non-identity elements order 2. Then G abelian, vector space over $\mathbb{Z}_2$. Choosing basis forces isomorphism $G \cong C_2 \times C_2 \cong K_4$. □

Proposition 3. *Let p prime. Then $\exists$ two groups of order $2p$ up to isomorphism, D_{2p}, C_{2p}.*

Proof. If $\exists\, g \in G$ with $|g| = 2p$ then $G \cong C_{2p}$ so suppose not.

If not must exist element of order p. (If only have $|g| = 1$ or 2 then pick x with $|x| = 2$. Hence $|G| = 2^n$ some $n \in \mathbb{N}$, but $2^n \neq 2p \quad \forall\, p > 2$.) Pick a with $|a| = p$. $H = \langle a \rangle \leq G$. $|H| = p$, so $|G : H| = \frac{|G|}{|H|} = 2$ so $H \trianglelefteq G$.

Now show $\exists\, b \in G \setminus H$ of order 2.

Pick $c \in G \setminus H$. H, Hc disjoint, so $G = H \cup Hc$. Consider what c^2 equals:

- $c^2 \in Hc \iff c^2 = a^j c$ some $0 \le j < p \iff c = a^j \in H$. This is contradiction as $c \notin H$.
- $c^2 = 1 \iff |c| = 2$. Let $b = c$.
- $c^2 \in H \setminus \{1\}$ then $c^2 = a^j$ some $1 \le j < p$ then $|c| = 2p$. Let $b = c^p = (c^2)^{\frac{p-1}{2}} c \in Hc \in G \setminus H$.

Consider ϕ, $h \mapsto bhb^{-1}$.

$$\phi^2(h) = b(bhb^{-1})b^{-1} = b^2h(b^{-1})^2 = b^2hb^2 = h$$

so $\phi^2 = 1$.

a and b generate G. $ba \notin H$ so $ba \in Hb$. $ba = a^j b \iff bab^{-1} = a^j$. In $\mathbb{Z}_p$ $x^2 = 1$ has solutions $x = 1$, $x = p - 1$ so either $\phi(a) = a$ or $\phi(a) = a^{p-1}$:

- $\phi(a) = a \Rightarrow ba = ab \Rightarrow G \cong C_2 \times C_p \cong c_{2p}$
- $\phi(a) = a^{p-1} \Rightarrow ba = a^{p-1}b = a^{-1}b \Rightarrow G \cong D_{2p}$ □

Proposition 4. $|G| = 8$. *Then G isomorphic to one of:*

- C_8
- $C_4 \times C_2$
- $C_2 \times C_2 \times C_2$
- D_8
- Q_8 *(quaternions)*

Proof. If $\exists\, g \in G$ s.t. $|g| = 8$ then $G \cong C_8$, so assume not.

If all non identity elements have order 2 then G is a vector space over $\mathbb{Z}_2$. Has dim 3 as $2^3 = 8$. Hence $G \cong C_2 \times C_2 \times C_2$. Otherwise $\exists\, a \in G$ with $|a| = 4$. $N = \langle a \rangle \trianglelefteq G$. Let $b \in G \setminus N$ so $G = N \cup Nb$. $b^2 \notin Nb$ so $b^2 \in N$.

If $b^2 = a$ or a^3 then $|b| = 8$, contradicting assumption above. Hence $b^2 = 1$ or a^2. N Normal so $bab^{-1} \in N$. $bab^{-1} \ne 1$.

If $bab^{-1} = a^2$ then $ba^2b^{-1} = (bab^{-1})(bab^{-1}) = a^2a^2 = 1$, then $a^2 = b^{-1}b = 1$. Contradiction as $|a| = 4$. Hence $bab^{-1} = a$ or a^3.

- $b^2 = 1,\ ba = ab \Rightarrow G \cong C_4 \times C_2$.
- $b^2 = a^2,\ ba = ab$. Then $(ab)^2 = a^4 = 1$ so replace b by ab and get $G \cong C_4 \times C_2$.
- $b^2 = 1,\ ba = a^{-1}b \Rightarrow G \cong D_8$.
- $b^2 = a^2,\ ba = a^{-1}b \Rightarrow G \cong Q_8$. □

Proposition 5. *p prime. There are two groups of order p^2 up to isomorphism: $C_p \times C_p$ and C_{p^2}.*

Proof. Non-isomorphic since C_{p^2} has element of order p^2.

Suppose $|G| = p^2$. By Lagrange $|Z(G)| = p$ or p^2.

- If $|Z(G)| = p^2$ then G abelian.
- Suppose $|Z(G)| = p$. Pick $x \in G \setminus Z(G)$, consider $\mathrm{C}_G(x)$. Contains $Z(G)$ and x, so $|\mathrm{C}_G(x)| > p$, so must be p^2. Hence $\mathrm{C}_G(x) = G$ and $x \in Z(G)$. Contradiction.

Therefore if $|G| = p^2$ then G abelian.

If $\exists\, a \in G$ with $|a| = p^2$ then $G \cong C_{p^2}$.

If not all non-identity elements have order p. Then G has vector space structure over $\mathbb{Z}_p$, so $G \cong C_p \times C_p$. □

2 Summary Table

Order	Form	Possible Isomorphisms	Ref
2	p	C_2	1
3	p	C_3	1
4	4	C_4, K_4	2
5	p	C_5	1
6	$2p$	C_6, D_6	3
7	p	C_7	1
8	8	C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, D_8, Q_8	4
9	p^2	C_9, $C_3 \times C_3$	5
10	$2p$	C_{10}, D_{10}	3
11	p	C_{11}	1