MA222
Introduction to Metric Spaces
and Toplogical Spaces

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Based on the lecture notes[1] of Professor David Preiss

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Contents

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[1] Available to download from `http://www.maths.warwick.ac.uk/~masfay/ma222`

1 Metric Spaces

Definition 1.1. A *metric* on set M is $d\colon M \times M \to \mathbb{R}$ s.t. $\forall\, x, y, z \in M$:

(M1) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$

(M2) $d(x,y) = d(y,x)$

(M3) $d(x,z) \leq d(x,y) + d(y,z)$.

Definition 1.2. The *open ball* centred at $a \in M$ with radius r:

$$B(a,r) = \{x \in M : d(x,a) < r\}$$

Proposition 1.1. *The ℓ_p norm $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ satisfies the triangle inequality.*

Proof. Let $\|x\|_p, \|y\|_p = 1$, $\alpha + \beta = 1$.

$B(0,1)$ is convex: $|\alpha x_i + \beta y_i|^p \leq \alpha |x_i|^p + \beta |y_i|^p$. Summing up gives $\|\alpha x + \beta y\|_p \leq 1$.

$\triangle$ inequality obvious if either x or $y = 0$ so assume not. Define $\hat{x}$, $\hat{y}$ as usual, $\lambda = \frac{1}{\|x\|_p + \|y\|_p}$, $\alpha = \lambda \|x\|_p$, $\beta = \lambda \|y\|_p$. Then $\|\hat{x}\|_p = \|\hat{y}\|_p = 1$, $\alpha, \beta, \lambda > 0$ and $\alpha + \beta = 1$, so $\|x + y\|_p = (\|x\|_p + \|y\|_p)\, \|\alpha \hat{x} + \beta \hat{y}\|_p \leq \|x\|_p + \|y\|_p$. ☐

Definition 1.3. $H \subset M$. Define $d_H(x,y) = d_M(x,y) \quad \forall\, x, y \in H$. Then (H, d_H) is a *(metric) subspace* of M.

If M_i are metric spaces, the product $M_1 \times \cdots \times M_n$ is a metric space with any of the metrics

$$d(x,y) = \left(\sum_{i=1}^{n} (d_i(x_i, y_i))^p \right)^{\frac{1}{p}} \qquad 1 \leq p \leq \infty$$

1.1 Open sets

Definition 1.4. $U \subseteq M$ open if $\forall\, x \in U\ \exists\, \delta > 0$ s.t. $B(x,\delta) \subset U$.

$F \subseteq M$ is closed if $M \setminus F$ is open.

Lemma 1.2. *Open balls are open.*

Proof. $x \in B(a,r)$. Let $\delta = r - d(x,a) > 0$. Let $y \in B(x,\delta)$.

$$\begin{aligned} d(y,a) &\leq d(y,x) + d(x,a) \\ &< \delta + d(x,a) \\ &= r \end{aligned}$$

☐

Proposition 1.3. *$U_1, \ldots, U_k \subset M$ open. Then $\bigcap_{i=1}^{k} U_i$ is open.*

Proof. $x \in \bigcap_{i=1}^k U_i$. Then for each i, $x \in U_i \Rightarrow \exists\, \delta_i > 0$ s.t. $B(x, \delta_i) \subset U_i$. Let $\delta = \min\{\delta_1, \dots, \delta_k\}$. Then $B(x, \delta) \subset \bigcap_{i=1}^k U_i$. □

Proposition 1.4. *Any union of open sets is open.*

Proof. Any point in the union is in one of the sets, this is open so exists open ball around in this set so in the union. □

1.2 Continuity

Definition 1.5. $f\colon M_1 \to M_2$ is

- *continuous at* $a \in M$ if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $\forall\, x \in M_1\ d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$.
- *continuous* if f cts at $a\ \forall\, a \in M_1$.
- *Lipschitz* if $\exists\, C \in \mathbb{R}$ s.t.

$$d_2(f(x), f(y)) \le C d_1(x, y) \quad \forall x, y \in M_1.$$

Theorem 1.5. *$f\colon M_1 \to M_2$ is cts iff $\forall\, U \subset M_2$ open $f^{-1}(U)$ is open (in M_1).*

Proof. ($\Rightarrow$) $x \in f^{-1}(U)$. $f(x) \in U$ so $B(f(x), \varepsilon) \subset U$ for some $\varepsilon > 0$. f cts at x so $\exists\, \delta > 0$ s.t. $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$.

($\Leftarrow$) Let $x \in M_1$, $\varepsilon > 0$. $B(f(x), \varepsilon)$ open so $f^{-1}(B(f(x), \varepsilon))$ open. $x \in f^{-1}(B(f(x), \varepsilon))$ so $\exists \delta > 0$ s.t. $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. Then $y \in B(x, \delta) \Rightarrow f(y) \in B(f(x), \varepsilon)$, so f cts at x. □

Definition 1.6. Two metrics d_1, d_2 on M are *topologically equivalent* if d_1 and d_2 open sets coincide.

Theorem 1.6. *If d_1, d_2 are two metrics on M then TFAE:*

1. *$\forall\, (N, d)$ every $f\colon M \to N$ is d_1 cts iff d_2 cts.*
2. *$\forall\, (N, d)$ every $g\colon N \to M$ is d_1 cts iff d_2 cts.*
3. *d_1 and d_2 are topologically equivalent.*

Proposition 1.7. *If $\exists\, 0 < c, C < \infty$ s.t. $\forall\, x, y \in M$*

$$c d_1(x, y) \le d_2(x, y) \le C d_1(x, y)$$

then d_1 and d_2 are topologically equivalent.

Proof. Let U be d_2 open, $x \in U$. Find $\delta > 0$ s.t. $B_{d_2}(x, \delta) \subset U$. Then $B_{d_1}\left(x, \frac{\delta}{C}\right) \subseteq B_{d_2}(x, \delta) \subset U$ so U is d_1 open. Converse similar. □

Definition 1.7. *Isometry* $f\colon M_1 \to M_2$ is bijection s.t. $d_2(f(x), f(y)) = d_1(x, y)$.

Homeomorphism $f\colon M_1 \to M_2$ is bijection s.t. U open in M_1 iff $f(U)$ open in M_2.

2 Topological spaces

Definition 2.1. *Topology* on a set T is a collection of open subsets $\mathcal{T}$ s.t.

- (T1) $\emptyset, T$ open
- (T2) any finite intersection of open sets open
- (T3) any union of open sets open.

Definition 2.2. $(T, \mathcal{T})$ a topological space, $S \subset T$ define $\mathcal{T}_S = \{(U \cap S) : U \in \mathcal{T}\}$. $(S, \mathcal{T}_S)$ called *(topological) subspace* of T.

2.1 Bases, sub-bases

Definition 2.3. A *basis* for $\mathcal{T}$ on T is $\mathcal{B} \subset \mathcal{T}$ s.t. every set from $\mathcal{T}$ is a union of sets from $\mathcal{B}$.

A *sub-basis* for $\mathcal{T}$ on T is $\mathcal{B} \subset \mathcal{T}$ s.t. every set from $\mathcal{T}$ is a union of finite intersections of sets from $\mathcal{B}$.

Example 1. Intervals (a, b) form basis for $\mathbb{R}$ with Euclidean topology, collection $(a, \infty), (-\infty, b)$ form sub-basis.

Proposition 2.1. *If $\mathcal{B}$ is any basis for $\mathcal{T}$ of T then*

- *(B1) T is a union of sets from $\mathcal{B}$.*
- *(B2) the intersection of any two sets from $\mathcal{B}$ is a union of sets from $\mathcal{B}$.*

Proposition 2.2. *If $\mathcal{B}$ is any collection of subsets of T satisfying (B1), (B2) then $\exists$ unique $\mathcal{T}$ with basis $\mathcal{B}$. Its open sets are unions of sets from $\mathcal{B}$.*

Proof. By defn of basis if such topology exists, it must be $\mathcal{T}$ the unions of sets from $\mathcal{B}$. RTP this is a topology.

- (T1) $\emptyset$ is union of no sets so $\emptyset \in \mathcal{T}$. By (B2) $T \in \mathcal{T}$.
- (T2) If $U = \bigcup_{i \in I} B_i$ and $V = \bigcup_{j \in J} D_j$ where $B_i, D_j \in \mathcal{B}$, then
$$U \cap V = \bigcup_{i,j} B_i \cap D_j$$
union of sets from $\mathcal{B}$ by (B2).
- (T3) Clear. ☐

Proposition 2.3. *If $\mathcal{B}$ any collection of subsets of T then $\exists$ uniqe topology on T with sub-basis $\mathcal{B}$. Open sets are unions of finite intersections of sets from $\mathcal{B}$.*

Proof. Let $\mathcal{D}$ be the collection of finite intersections from $\mathcal{B}$. Any topology with sub-basis $\mathcal{B}$ has basis $\mathcal{D}$. $\mathcal{D}$ satisfies (B1),(B2) so $\exists! \mathcal{T}$ with basis $\mathcal{D}$. ☐

Definition 2.4. $(T_1, \mathcal{T}_1), (T_2, \mathcal{T}_2)$ topological spaces, *product topology* on $T_1 \times T_2$ has basis
$$\mathcal{B} = \{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}$$

2.2 Continuity

Definition 2.5. $f\colon T_1 \to T_2$ is *continuous* if $\forall U \subset T_2$ open $f^{-1}(U)$ open in T_1.

Theorem 2.4. *$f\colon T_1 \to T_2$ and $g\colon T_2 \to T_3$ cts then $g \circ f\colon T_1 \to T_3$ cts.*

Proof. U open in T_3 so $g^{-1}(U)$ open in T_2 as g cts. Then $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right)$ open in T_1 as f cts. ☐

Proposition 2.5. Projection $\pi_1\colon T_1 \times T_2 \to T_1$, $\pi_1(x, y) = x$ *(and similarly π_2) is continuous.*

Proof. U_1 open in T_1, $\pi_{-1}(U_1) = U_1 \times T_2$ open in $T_1 \times T_2$. ☐

Lemma 2.6. *T_1, T_2 topological spaces, $\mathcal{B}$ sub-basis for T_2. $f\colon T_1 \to t_2$ cts iff $f^{-1}(B)$ open $\forall\, B \in \mathcal{B}$.*

Proof. Preimages preserve unions and intersections. ☐

Proposition 2.7. *$f\colon T \to T_1 \times T_2$ is cts iff f_1 and f_2 are cts.*

Proof. ($\Rightarrow$) π_i cts so $f_i = \pi_i \circ f$ cts.

($\Leftarrow$) If U_i open in T_i, $f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$ open in T, so cts by lemma 2.6. ☐

Definition 2.6. Bijection $f\colon T_1 \to T_2$ *homeomorphism* if U open in $T_1 \iff f(U)$ open in T_2.

So a homeomorphism is a cts bijection with cts inverse.

2.3 Closure, interior, boundary

Definition 2.7.

- *Interior* H° of $H \subset T$ is union of all sets open in T contained in H.
- *Neighbourhood* of x is H s.t. $x \in H^\circ$.
- *Closure* $\overline{H}$ of H is set of x s.t. every neighbourhood of x meets H.

Lemma 2.8. *$H^\circ = T \setminus \overline{(T \setminus H)}$ and $\overline{H} = T \setminus (T \setminus H)^\circ$.*

Proof. $x \in H^\circ$, H neighbourhood of x not meeting $T \setminus H$ so $x \notin \overline{(T \setminus H)}$, so $x \in T \setminus \overline{(T \setminus H)}$.

If $x \in T \setminus \overline{(T \setminus H)}$, $x \notin \overline{(T \setminus H)}$ so exists neighbourhood of x not meeting $T \setminus H$. This neighbourhood must be contained in H so $x \in H^\circ$. ☐

From this we get:

- H° open (largest open subset of H), $\overline{H}$ closed (least closed superset of H).

- H open iff $H = H^\circ$, closed iff $H = \overline{H}$.
- $(H^\circ)^\circ = H^\circ$, $\overline{\overline{H}} = \overline{H}$.
- $H \subset K \Rightarrow H^\circ \subset K^\circ$, $\overline{H} \subset \overline{K}$.
- $(H \cap K)^\circ = H^\circ \cap K^\circ$, $\overline{H \cup K} = \overline{H} \cup \overline{K}$.

Definition 2.8. *Boundary* ∂H of H is set of points x whose every neighbourhood meets H and $T \setminus H$.

Proposition 2.9. *$\partial H = \overline{H} \cap \overline{(T \setminus H)}$ is closed.*

2.4 Hausdorff spaces

Definition 2.9. Topological space T is *Hausdorff* if $\forall\, x \neq y \in T \quad \exists$ disjoint open sets U, V containing x, y respectively.

Proposition 2.10. *Every metric space is Hausdorff.*

Proof. Let $r = d(x, y) > 0$. Then $B\left(x, \frac{r}{2}\right), B\left(y, \frac{r}{2}\right)$ are disjoint. ☐

3 Compactness

Definition 3.1.
- A *cover* of A is a collection $\mathcal{U}$ of sets whose union contains A.
- A *subcover* is a subcollection of $\mathcal{U}$ which still covers A.
- A cover is *open* if its members are all open.

Definition 3.2. Topological space T is *compact* if every open cover has a finite subcover.

Theorem 3.1 (Heine-Borel)**.** *Any closed bounded interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. Let $\mathcal{U}$ be open cover of $[a, b]$. Let

$$A = \{x \in [a, b] : [a, x] \text{ covered by finite subfamily of } \mathcal{U}\}$$

Then $a \in A$ so $A \neq \emptyset$, bounded above by b. Let $c = \sup A$. $a \leq c \leq b$ so $c \in U$ for some $U \in \mathcal{U}$. U open so $\exists\, \delta > 0$ s.t. $(c - \delta, c + \delta) \subset U$.

$c = \sup A$ so $\exists\, x \in A$ s.t. $x > c - \delta$. $[a, c + \delta) \subseteq [a, x] \cup (c - \delta, c + \delta)$ can be covered by finite subfamily of $\mathcal{U}$ so $(c, c + \delta) \cap [a, b] = \emptyset$ (since any point in here is in A but $> c = \sup A$). So $c = b$. ☐

3.1 Compactness of subsets

Proposition 3.2. *Any closed subset C of compact space compact.*

Proof. Let $\mathcal{U}$ be cover of C by sets open in T. Adding open $T \setminus C$ get open cover of T. Finite subcover of this cover contains finite subcover of C of sets from $\mathcal{U}$. □

Proposition 3.3. *Compact subspace C of Hausdorff T is closed in T.*

Proof. $a \in T \setminus C$. $\forall\, x \in C \quad \exists$ disjoint $U_x \ni x, V_x \ni a$ open in T since T Hausdorff. U_x open cover of C so has finite subcover $U_{x_1}, \dots, U_{x_n}$. Then $V = \bigcap_{i=1}^{n} V_{x_i}$ open, $a \in V$ and disjoint from C. Hence $a \in (T \setminus C)^{\circ}$ and $T \setminus C$ open. □

Proposition 3.4. *Compact subspace C of metric space M is bounded.*

Proof. Let $a \in M$. Balls $B(a, r) \quad (r > 0)$ are open and cover C, so $\exists\, r_1, \dots, r_n$ s.t. $C \subset \bigcup_{i=1}^{n} B(a, r_i) = B(a, \max\{r_1, \dots, r_n\})$. □

3.2 Intersections of closed sets

Theorem 3.5. *Let $\mathcal{F}$ be collection of non-empty closed subsets of compact T s.t. every finite subcollection of $\mathcal{F}$ has non-empty intersection. Then intersection of all sets from $\mathcal{F}$ non-empty.*

Proof. Assume intersection of all sets empty. Let $\mathcal{U}$ be collection of complements. $\mathcal{U}$ covers T by De Morgan. $\mathcal{U}$ open cover so exists finite subcover $U_1, \dots, U_n$. Then $F_i := T \setminus U_i \in \mathcal{F}$ and empty intersection by De Morgan. This contradicts the assumption of the theorem. □

Corollary 3.6. *Let $F_1 \supset F_2 \supset \dots$ sequence of non-empty closed subsets of compact T. Then $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$.*

Corollary 3.7. *Let $F_1 \supset F_2 \supset \dots$ sequence of non-empty compact subsets of Hausdorff T. Then $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$.*

Proof. By proposition 3.3 compact subsets of Hausdorff space are closed. □

3.3 Compactness of products

Lemma 3.8. *T, S compact, $\mathcal{U}$ open cover of $T \times S$. If $s \in S$ there exists open $V \subset S$, $s \in V$ s.t. $T \times V$ can be covered by finite subfamily of $\mathcal{U}$.*

Proof. $\forall\, x \in T$ find $W_x \in \mathcal{U}$ s.t. $(x,s) \in W_x$. Exists open $U_x \subset T, \quad V_x \subset S$ s.t. $(x,s) \in U_x \times V_x \subset W_x$. $\{U_x : x \in T\}$ open cover of T so $\exists\, U_{x_1}, \dots, U_{x_n}$ which cover T. Let $V = \bigcap_{i=1}^n V_{x_i}$. $V \subset S$ open and

$$T \times V \subset \bigcup_{i=1}^n U_{x_i} \times V_{x_i} \subset \bigcup_{i=1}^n W_{x_i} \qquad \square$$

Theorem 3.9 (Tychonov)**.** *S, T compact $\Rightarrow$ $T \times S$ compact.*

Proof. By lemma 3.8 $\forall\, y \in S\ \exists\, V_y \subset S$ open s.t. $T \times V_y$ can be covered by finite subfamily of $\mathcal{U}$. S compact, $\{V_y : y \in S\}$ form open cover so $\exists\ V_{y_1}, \dots, V_{y_m}$ which cover S.

$T \times S = \bigcup_{j=1}^m T \times V_{y_j}$. Finite union, each $T \times V_{y_j}$ can be covered by finite subfamily of $\mathcal{U}$, so $T \times S$ can be covered by finite subfamily of $\mathcal{U}$. ☐

3.4 Compactness and continuity

Proposition 3.10. *Cts image of compact space compact.*

Proof. $f\colon T \to S$ cts, T compact. $\mathcal{U}$ open cover of $f(T)$. $f^{-1}(U)$ open $\forall\, U \in \mathcal{U}$. Cover T since $\forall\, x \in T$ $f(x)$ in some $U \in \mathcal{U}$. Hence $\exists\ f^{-1}(U_1), \dots, f^{-1}(U_n)$ subcover of T. $\forall\, y \in f(T)$ have $y = f(x)$ where $x \in T$ so $x \in f^{-1}(U_i)$ for some i so $y \in U_i$. Hence $U_1, \dots, U_n$. ☐

Theorem 3.11. *Cts bijection of compact T onto Hausdorff S is homeomorphism.*

Proof. U open in T, $T \setminus U$ closed so compact by theorem 3.2. Hence $f(T \setminus U)$ compact by 3.10 so closed by 3.3.

Therefore $(f^{-1})^{-1}(U) = f(U) = S \setminus f(T \setminus U)$ open, so f^{-1} cts. ☐

Corollary 3.12. *Let T be compact. Cts $f\colon T \to \mathbb{R}$ is bdd and attains max and min.*

Proof. $f(T)$ compact by 3.10 so closed and bdd by 3.3 and 3.4.

Then $\sup f(T) \in \overline{f(T)} = f(T)$. ☐

Alternative proof. Let $c = \sup_{x \in T} f(x)$. If f not attain c then $f(x) < c\ \forall\, x$ so $\{x : f(x) < r\} = f^{-1}(-\infty, a)$ where $r < c$ s.t. $T \subset \bigcup_{i=1}^n \{x : f(x) < r_i\}$. Then $f(x) < \max\{r_1, \dots, r_n\} \forall\, x$ so $c = \sup_{x \in T} f(x) \le \max\{r_1, \dots, r_n\} < c$. Contradiction. ☐

Definition 3.3. Given cover $\mathcal{U}$ of metric M, $\delta > 0$ called *Lebesgue number of* $\mathcal{U}$ if $\forall\, x \in M\ \exists\, U \in \mathcal{U}$ s.t. $B(x, \delta) \subset U$.

Proposition 3.13. *Every open cover $\mathcal{U}$ of compact metric space has a Lebesgue number.*

Proof. $\forall\, x \in M$ pick $r(x) > 0$ s.t. $B(x, r(x))$ contained in some set of $\mathcal{U}$. Then $M \subset \bigcup_{x\in M} B\left(x, \frac{r(x)}{2}\right)$ so $\exists\, x_1, \ldots, x_j$ s.t. $M \subset \bigcup_{i=1}^{j} B\left(x_i, \frac{r(x_i)}{2}\right)$. Let $\delta = \frac{\min\{r(x_1),\ldots,r(x_j)\}}{2}$. Then $\forall\, x \in M$ pick i s.t. $x \in B\left(x_i, \frac{r(x_i)}{2}\right)$ and $B(x, \delta) \subset B(x_i, r(x_i))$ subset of some set from $\mathcal{U}$. ☐

Theorem 3.14. *Cts map of compact metric M to metric N is uniformly cts.*

Proof. Let $\varepsilon > 0$. Then sets $U_z = f^{-1}\left(B_N\left(f(z), \frac{\varepsilon}{2}\right)\right)$ $z \in M$ open cover of M. Let δ be Lebesgue number. If $x, y \in M$, $d_M(x, y) < \delta \Rightarrow\ y \in B(x, \delta) \subset U_z$ some z so $d_N(f(x), f(y)) \leq d_N(f(x), z) + d_N(f(y), z) < \varepsilon$. ☐

3.5 Compact sets in $\mathbb{R}^n$

Theorem 3.15 (Heine-Borel). *$A \subset \mathbb{R}^n$ compact iff closed and bdd.*

Proof. ($\Rightarrow$) Metric spaces are Hausdorff, so A closed and bdd by 3.3 and 3.4.

($\Leftarrow$) $C \subset \mathbb{R}^n$ bdd $\Rightarrow \exists\, [a, b] \subset \mathbb{R}^n$ s.t. $C \subset [a, b] \times \cdots \times [a, b]$. This compact by Tychanov (3.9). If C closed then closed subset of compact space so compact by 3.2. ☐

3.6 Sequential compactness

Theorem 3.16. *Metric M is compact iff every sequence in M has convergent subsequence.*

Lemma 3.17. *A_k sequence of subsets of metric M. Then $\forall\, x \in \bigcap_{j=1}^{\infty} \overline{A_j}$ $\exists\, x_k \in A_k$ s.t. $x_k \to x$.*

Proof. Take $x_k \in A_k \cap B\left(x, \frac{1}{k}\right) \neq \emptyset$. ☐

Corollary 3.18. *$x_k \in M$ and $\bigcap_{j=1}^{\infty} \overline{\{x_j, x_{j+1}, \ldots\}} \neq \emptyset$ then x_k have convergent subsequence.*

Proof. Let $x \in \bigcap_{j=1}^{\infty} \overline{\{x_j, x_{j+1}, \ldots\}}$. By lemma 3.17 $\exists\, k_j \geq j$ s.t. $x_{k_j} \to x$. $k_j \to \infty$ so can choose subsequence k_{j_i} s.t. $k_{j_{i+1}} > k_{j_i}$ (as k_js not necessarily in order). Then $x_{k_{j_i}}$ subsequence converging to x. ☐

Proof of ($\Rightarrow$) of theorem 3.16. Let $x_k \in M$, $F_j = \overline{\{x_j, x_{j+1}, \ldots\}}$. F_j form decreasing sequence of non-empty closed subsets of M.

By corollary 3.6 $\bigcap_{j=1}^{\infty} F_j \neq \emptyset$ so x_k have convergent subsequence by corollary 3.18. ☐

Notation:

$\mathcal{U}$ open cover of M. $\forall\, x \in M$

$$r(x) = \sup\{r \leq 1 : \exists\, U \in \mathcal{U} \text{ s.t. } B(x,r) \subset U\}$$

Lemma 3.19. *If $y_k \to x$ $\exists\, K$ s.t. $y_{k+1} \in B\left(y_k, \frac{r(y_k)}{2}\right)$ for $k \geq K$.*

Proof. Let $U \in \mathcal{U}$ be s.t. $B\left(x, \frac{r(x)}{2}\right) \subset U$. Take K s.t. $d(y_k, x) < \frac{r(x)}{16}$ for $k \geq K$. Then $k \geq K \Rightarrow B\left(y_k, \frac{r(x)}{2} - d(x, y_k)\right) \subset B\left(x, \frac{r(x)}{2}\right) \subset U$, so $r(y_k) \geq \frac{r(x)}{2} - d(x, y_k) \geq \frac{r(x)}{4}$, so

$$d(y_{k+1}, y_k) \leq d(y_{k+1}, x) + d(y_k, x) < \frac{r(x)}{8} \leq \frac{r(y_k)}{2}$$ □

Proof of ($\Leftarrow$) of theorem 3.16. IDEA: Greedy Algorithm

$M_1 := M$, $s_1 := \sup_{x \in M_1} r(x)$. Find $x_1 \in M_1$ s.t. $r(x_1) > \frac{s_1}{2}$, choose $U_1 \in \mathcal{U}$ s.t. $B\left(x_1, \frac{r(x_1)}{2}\right) \subset U_1$.

If $x_1, \ldots, x_j$ have been defined,

$$M_{j+1} := M \setminus B\left(x_j, \frac{r(x_j)}{2}\right) = M \setminus \bigcup_{i=1}^{j} B\left(x_i, \frac{r(x_i)}{2}\right)$$

If $M_{j+1} = \emptyset$ then $M \subset \bigcup_{i=1}^{j} B\left(x_i, \frac{r(x_i)}{2}\right) \subset \bigcup_{i=1}^{j} U_i$ has finite subcover.

If $M_{j+1} \neq \emptyset$ let $s_{j+1} = \sup_{x \in M_{j+1}}\{r(x)\}$, find x_{j+1} s.t. $r(x_{j+1}) > \frac{s_{j+1}}{2}$, choose $U_{j+1} \in \mathcal{U}$ s.t. $B\left(x_{j+1}, \frac{r(x_{j+1})}{2}\right) \subset U_{j+1}$.

If procedure stops we have finite subcover. If it runs forever we have infinite sequence x_j s.t. $x_i \notin B\left(x_j, \frac{r(x_j)}{2}\right)$ for $i > j$. This has cvgt subsequence x_{j_k} by assumption, so by lemma 3.19 $\exists\, k$ s.t. $B\left(x_{j_k}, \frac{r(x_{j_k})}{2}\right)$. This is a contradiction, so the procedure stops. □

4 Connectedness

4.1 Connected, separated

Definition 4.1. Topological T *connected* if for every decomposition $T = A \cup B$ into disjoint open A, B either A or B is empty.

Definition 4.2. $T \subset S$ *separated* by sets $U, V \subset S$ if $T \subset U \cup V$, $U \cap V \cap T = \emptyset$, $U \cap T \neq \emptyset$, $V \cap T \neq \emptyset$.

Proposition 4.1. *$T \subset S$ disconnected iff T is separated by some $U, V \subset S$.*

Proof. (⇒) If disconnected $\exists A, B \subset T,\ A, B \neq \emptyset$ s.t. $T = A \cup B$ and $A \cap B = \emptyset$. $T \subset S$ so $\exists U, V$ open in S s.t. $A = U \cap T$, $B = V \cap T$. Then U, V separate T.

(⇐) If U, V separate T let $A = U \cap T$, $B = V \cap T$ then T not connected. □

Proposition 4.2. *TFAE:*

1. *T disconnected*
2. *T has subset which is open, closed, different from $\emptyset, T$*
3. *T admits non-constant cts function to two point discrete space.*

Proof.

(1. ⇒ 2.) $\exists$ decomposition $T = A \cup B$ with A, B open, non-empty. Hence $A = T \setminus B$ is open and closed, different from $\emptyset, T$.

(2. ⇒ 3.) $\emptyset, T \neq A \subset T$ open, closed. Define $f\colon T \to \{0,1\}$ by $f(x) = \begin{cases} 0 & x \in A \\ 1 & x \notin A \end{cases}$

This cts as preimages open.

(3. ⇒ 1.) $f\colon T \to \{0,1\}$ non-constant and cts. Define $A = F^{-1}(0)$, $B = f^{-1}(1)$. □

4.2 Connectedness in metric spaces

Theorem 4.3. *$T \subset M$ (M metric) disconnected iff $\exists$ disjoint open $U, V \subset M$ s.t. $T \cap U \neq \emptyset \neq T \cap V$ and $T \subset U \cup V$.*

Proof. (⇐) Clear

(⇒) $T = A \cup B$. Let

$$U = \{x \in M : d(x, A) < d(x, B)\}$$
$$V = \{x \in M : d(x, A) > d(x, B)\}$$

U, V disjoint, open.

Going to prove $A \subset U$: Let $x \in A$. A open in T so $\exists\, \delta > 0$ s.t. $B(x, \delta) \cap T \subset A$. $B \subset T$ disjoint from A so $B(x, \delta) \cap B = \emptyset$, so $d(x, B) \geq \delta > 0$. Since $d(x, A) = 0$ we have $x \in U$. Similarly $B \subset V$. □

Lemma 4.4. *$I \subset \mathbb{R}$ is an interval iff $\forall\, x, y \in I$, $\forall\, z \in \mathbb{R}$,*

$$x < z < y \Rightarrow z \in I$$

Proof. Intervals clearly have this property. Conversely suppose I has above property, non-empty, not single point. Let $a = \inf I$, $b = \sup I$.

Show $(a, b) \subset I$: If $z \in (a, b)$ $\exists\, x, y \in I$ with $x < z < y$ so $z \in I$. Hence $(a, b) \subset I \subset (a, b) \cup \{a, b\}$. ☐

Theorem 4.5. *$T \subset \mathbb{R}$ connected iff it is an interval.*

Proof. ($\Rightarrow$) Suppose I not interval. Then by lemma 4.4 $\exists\, x, y \in I,\ z \in \mathbb{R}$ s.t. $x < z < y$ and $z \notin I$. Let $A = (-\infty, z) \cap I$, $B = (z, \infty) \cap I$. A, B disjoint, non-empty, open and $I = A \cup B$.

($\Leftarrow$) Suppose I not connected. Then $\exists$ cts non-constant $f\colon I \to \{0, 1\}$ where $\{0, 1\}$ has discrete topology. But then f also cts as funct $f\colon I \to \mathbb{R}$ contradicting IVT.

($\Leftarrow$) I partitioned into non-empty A, B open. Choose $a \in A$, $b \in B$, $a < b$. A, B open cover of $[a, b]$. Let δ be its Lebesgue number. Then $\left[a, a + \frac{\delta}{2}\right] \subset A$, $\left[a + \frac{\delta}{2}, a + \frac{2\delta}{2}\right] \subset A, \ldots$ until we get to an interval containing b. So $b \in A$ and A, B not disjoint. ☐

4.3 Connected spaces from others

Proposition 4.6. *Cts image of connected space connected.*

Proof. Suppose $f\colon T \to S$ cts, T connected. If $f(T)$ disconnected $\exists\, U, V \subset S$ open separating $f(T)$. Then $f^{-1}(U), f^{-1}(V)$ open, disjoint, cover T. Contradiction as T connected. ☐

Proposition 4.7. *If $C, C_j \quad (j \in J)$ connected subspaces of topological T and if $c_j \cap \overline{C} \neq \emptyset \quad \forall\, j \in J$ then*

$$K = C \cup \bigcup_{j \in J} C_j$$

is connected.

Proof. Suppose K disconnected. Hence $\exists\, U, V \subset T$ open that separate K.

C connected so cannot be separated by U, V, so does not meet one of them. Suppose w.l.o.g $C \cap V = \emptyset$. Then $C \subset U$. Since V open $\overline{C} \cap V = \emptyset$, so $K \cap \overline{C} \subset U$. Then $C_j \cap U \neq \emptyset \quad \forall\, j$.

C_j connected so $C_j \subset U$ or $C_j \subset V$. $C_j \cap U \neq \emptyset$ so $C_j \subset U$.

Then $K \subset U$ contradicting $V \cap K \neq \emptyset$. ☐

Corollary 4.8. *$C \subset T$ connected and $C \subset K \subset \overline{C}$. Then K connected.*

Proof. $K = C \bigcup_{x \in K} \{x\}$ and $\{x\} \cap \overline{C} \neq \emptyset \quad \forall\, x$. ☐

Proposition 4.9. *Product of connected spaces is connected.*

Proof. Let T, S connected, $s_0 \in S$. Define $C = T \times \{s_0\}$ and $C_t = \{t\} \times S$ (for some $t \in T$). Then C, C_t homeomorphic to T and S are connected. $C_t \cap C \neq \emptyset$ and $T \times S = C \cup \bigcup_{t \in T} C_t$ connected by 4.7. □

Example 2.

$$\sin\left(\frac{1}{t}\right) \cup \underbrace{\{(0,t) \in \mathbb{R}^2 : t \in (-1,1)\}}_{I}$$

is connected.

Proof.

$$C = \left\{(t, \sin\left(\frac{1}{t}\right)) : t > 0\right\}$$
$$D = \left\{(t, \sin\left(\frac{1}{t}\right)) : t < 0\right\}$$

C, D, I cts images of intervals so connected.

$(0,0) \in I$ is in $\overline{C}$ as $\left(t_k, \sin\left(\frac{1}{t_k}\right)\right) \to (0,0)$ when $t_k = \frac{1}{k\pi}$. Then $I \cup C$ connected by 4.7. Similarly $I \cup D$. □

4.4 Connected components

Definition 4.3. $x \sim y$ if x, y belong to a common connected subspace of T. Equivalence classes are *connected components* of T.

Are maximal connected subsets of T. Number of connected components is topological invariant.

Property $T \setminus \{x\}$ connected $\forall\, x \in T$ topological invariant.

4.5 Path connectedness

Definition 4.4. $a, b \in T$. $\varphi\colon [0,1] \to T$ cts with $\varphi(0) = a$, $\varphi(1) = b$ called a *path* from a to b.

Definition 4.5. T *path connected* if any two points can be joined by a path.

Proposition 4.10. *Path connected $\Rightarrow$ connected.*

Proof. $a \in T$. $\forall\, x \in T$ image C_x of path a to x is connected, and all C_x contain a. Then $T = \bigcup_{x \in T} C_x$ connected by 4.7. □

4.6 Open sets in $\mathbb{R}^n$

Theorem 4.11. *Any $U \subset \mathbb{R}^n$ open, connected is path connected.*

Proof. Let $a \in U$, $V = \{x \in U : \exists \text{ path from } a \text{ to } x\}$.

Let $z \in U \cap \overline{V}$. Find $\delta > 0$ s.t. $B(z,\delta) \subset U$. $z \in \overline{V}$ so $\exists\, y \in V \cap B(x,\delta)$.

Then $B(z,\delta) \subset V$ since join path from a to y to path from y to z. □

Theorem 4.12. *All components of open $U \subset \mathbb{R}^n$ open.*

Proof. C component of U, $x \in C$. Find $\delta > 0$ with $B(x,\delta) \subset U$. $B(x,\delta)$ connected and C union of all connected subsets of U containing x so $B(x,\delta) \subset C$, so C open. □

Theorem 4.13. *$U \subset \mathbb{R}$ open iff disjoint union of countably many open intervals.*

Proof. ($\Leftarrow$) Any union of open sets open.

($\Rightarrow$) $U \subset \mathbb{R}$ open, $C_j \quad (j \in J)$ its components. C_j connected and open so are open intervals by 4.5.

For each j $\exists$ rational $r_j \in C_j$. C_j^{s} mutually disjoint so $j \to r_j$ injection into $\mathbb{Q}$, so can order J into a sequence. □

5 Completeness

This is a concept that makes sense in metric spaces only.

Recall Cauchy.

Definition 5.1. Metric M is *complete* if every Cauchy sequence in M converges (to a point of M).

Remark 1. This is not a topological invariant: $(0,1)$ - incomplete and $\mathbb{R}$ - complete are homeomorphic.

Proposition 5.1. *Cvgt $\Rightarrow$ Cauchy.*

Proof. $\forall\, \varepsilon > 0\ \exists\, N$ s.t. $d(x_n,x) < \frac{\varepsilon}{2}$ for $n \geq N$. If $m,n \geq N$ then

$$d(x_m,x_n) \leq d(x_m,x) + d(x_n,x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ □

Proposition 5.2. *1. Complete subspace S of metric M is closed.*

2. Closed subset S of complete M is complete.

Proof. 1. Let $x_n \in S$, $x_n \to x \in M$. (x_n) Cauchy in S so cvgs in S to $y \in S$. $S \leq M$ so $x_n \to y$ in M. By uniqueness of limits $x = y \in S$.

2. Let $(x_n) \subset S$ Cauchy. Cauchy in M so cvgs to point of M which in S as S closed. $\square$

Proposition 5.3. *$\forall\, S$ $\mathcal{B}(S)$ of bdd functions $S \to \mathbb{R}$ with sup norm is complete.*

Proof. Let (f_n) Cauchy, $\varepsilon > 0$. $\exists\, N$ s.t. $\|f_m - f_n\| < \varepsilon$ for $n, m \geq N$. Hence for fixed x $(f_n(x))$ Cauchy in $\mathbb{R}$, so cvgs to $f(x) \in \mathbb{R}$.

For $n \geq N$ $|f_m(x) - f_n(x)| < \varepsilon \quad \forall\, m \geq N$. Let $m \to \infty$ then

$$|f(x) - f_n(x)| \leq \varepsilon \quad \forall\, x \in S,\ n \geq N$$

Then f bdd and $f_n \to f$. $\square$

5.1 Proving Cauchy

Proposition 5.4. *A sequence $(x_n) \subset M$ is Cauchy iff $\exists$ sequence $\varepsilon_n \geq 0$ s.t. $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$ and $d(x_m, x_n) \leq \varepsilon_n$ for $m > n$.*

Proof. $(\Rightarrow)$ Suppose (x_n) Cauchy. Then let $\varepsilon_n = \sum_{m>n} d(x_m, x_n) \to \underset{n\to\infty}{\longrightarrow} 0$.

$(\Leftarrow)$ Given $\varepsilon > 0$ find k s.t. $\varepsilon_n < \varepsilon$ for $n \geq k$. Then $d(x_n, x_n) \leq \varepsilon_n < \varepsilon$ for $m > n \geq k$. Exchanging m, n gives $d(x_m, x_n) < \varepsilon\ \forall\, n, m \geq k$. $\square$

Proposition 5.5. *$(x_n) \subset M$ sequence s.t. $\exists\, \tau_n \geq 0$ with $\sum_{n=1}^{\infty} \tau_n < \infty$ and $d(x_n, x_{n+1}) \leq \tau_n \quad \forall\, n$. Then (x_n) is Cauchy.*

Proof. Follows from 5.4 with $\varepsilon_n = \sum_{k=n}^{\infty} \tau_n$. Then

$$d(x_m, x_n) \underset{\triangle\text{ ineq}}{\leq} \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \tau_n \leq \varepsilon_n \qquad \square$$

5.2 Examples

Example 3. If K compact topological space then space $C(K)$ with sup norm is complete.

Proof. Each f bdd, attains max. By 5.2 suffices to show $C(K)$ closed in $\mathcal{B}(K)$.

Suppose $f_n \in C(K)$ cvg to $f \in \mathcal{B}(K)$. Then $\forall\, \varepsilon > 0\ \exists\, N$ s.t.

$$\sup_{x\in K} |f(x) - f_n(x)| < \varepsilon \quad \forall n \geq N$$

$$\forall\, a \in \mathbb{R} \quad \{x : f(x) > a\} = \bigcup_{\varepsilon>0} \{x : f_N(x) > a + \varepsilon\}$$

RHS are preimages of open sets so open. Hence LHS is open. Similarly $\{x : f(x) < a\}$ open. $(-\infty, a), (a, \infty)$ form sub-basis for $\mathbb{R}$ so f cts. $\square$

Example 4. $C[0,1]$ with norm $\|f\|_1 = \int_0^1 |f(x)|dx$ is incomplete.

Proof.

$$f_n(x) = \begin{cases} \min\left\{\sqrt{n}, \frac{1}{\sqrt{x}}\right\} & x > 0 \\ \sqrt{n} & x = 0 \end{cases}$$

so $(f_n) \subset C[0,1]$.

$$\begin{aligned} \int_0^1 |f_m(x) - f_n(x)|dx &= \int_0^{\frac{1}{m}} (\sqrt{m} - \sqrt{n})dx + \int_{\frac{1}{m}}^{\frac{1}{n}} \left(\frac{1}{\sqrt{x}} - \sqrt{n}\right) dx \\ &\leq \frac{1}{\sqrt{m}} + \frac{2}{\sqrt{n}} \\ &\leq \frac{3}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

so (f_n) Cauchy.

Let $f \in C[0,1]$. Find $k \in \mathbb{N}$ s.t. $|f| \leq \sqrt{k}$. Then for $n > k$

$$\begin{aligned} \int_0^1 |f_n(x) - f(x)|dx &\geq \int_{\frac{1}{n}}^{\frac{1}{m}} \left(\frac{1}{\sqrt{x}} - f(x)\right) dx \\ &\geq 2\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{n}}\right) - \frac{1}{\sqrt{k}} \\ &= \frac{1}{\sqrt{k}} - \frac{2}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{k}} > 0 \end{aligned}$$

☐

5.3 Completion

Definition 5.2. $S \subset M$ is *dense* in M if $\overline{S} = M$.

Definition 5.3. A *completion* of metric space M is:

- complete metric space N s.t. M dense subset of N.
- complete metric space N and isometry $i\colon M \to A \subseteq N$ s.t $i(M)$ is dense in N.

Theorem 5.6. *Any metric M can be isometrically embedded into complete metric space.*

Proof. Find isometry of M onto subset of $\mathcal{B}(M)$, complete by 5.3. Fix $a \in M$, define $F\colon M \to \mathcal{B}(M)$ by $F(x)(z) = d(z,x) - d(z,a)$. $|F(x)(z)| \leq d(x,a)$ so $F(x) \in \mathcal{B}(M)$.

$$\begin{aligned} |F(x)(z) - F(y)(z)| &= |d(z,x) - d(z,y)| \\ &\leq d(x,y) \end{aligned}$$

Equality occurs when $z = y$. Then $\|F(x) - F(y)\| = d(x,y)$ so F isometry. ☐

Corollary 5.7. *Any metric space M has a completion.*

Proof. Embet M into complete N. Then $\overline{M}$ (closure taken in N) is complete by 5.2, M dense in $\overline{M}$. Then $\overline{M}$ completion of M. ☐

5.4 Contraction Mapping Theorem

Definition 5.4. $f\colon M \to M$ *contraction* if $\exists\, \kappa < 1$ s.t.

$$d(f(x), f(y)) \leq \kappa d(x, y) \quad \forall\, x, y \in M$$

Theorem 5.8 (Banach)**.** *If f contraction on complete metric M then f has unique fixed point.*

Proof. Uniqueness: If $f(x) = x,\ f(y) = y$ then

$$d(x, y) = d(f(x), f(y)) \leq \kappa d(x, y) \Longrightarrow d(x, y) = 0$$

Existence: Pick $x_0 \in M$, $x_{n+1} = f(x_n)$. By repeated application of the contraction property we get that $d(x_j, x_{j+1}) \leq \kappa^j d(x_0, x_1)$. $\sum_{j=1}^{\infty} \kappa^j d(x_0, x_1) < \infty$ so (x_n) Cauchy by 5.5.

M complete so $x_n \to x \in M$, so $f(x_n) \to f(x)$. But also $f(x_n) = x_{n+1} \to x$ so $f(x) = x$. $\square$

5.5 Total boundedness

Definition 5.5. Metric M *totally bounded* if $\forall\, \varepsilon > 0\ \exists$ finite set $F \subset M$ s.t. $M \subset \bigcup_{x \in F} B(x, \varepsilon)$.

Proposition 5.9. *Subspace M of metric N is totally bounded iff $\forall\, \varepsilon > 0\ \exists$ finite $H \subset N$ s.t. $M \subset \bigcup_{z \in H} B(z, \varepsilon)$.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Given $\varepsilon > 0$ let $H \subset N$ be finite set s.t. $M \subset \bigcup_{z \in H} B\left(z, \frac{\varepsilon}{2}\right)$. From each non-empty $M \cap B\left(z, \frac{\varepsilon}{2}\right)$ pick one point. Let F be set of these points. $F \subset M$ finite.

If $y \in M$ then y in one of $B\left(z, \frac{\varepsilon}{2}\right)$ so $M \cap B\left(z, \frac{\varepsilon}{2}\right) \neq \emptyset$ so $\exists\, x \in M \cap B\left(z, \frac{\varepsilon}{2}\right)$. Hence $y \in B(x, \varepsilon)$ and $M \subset \bigcup_{x \in F} B(x, \varepsilon)$. $\square$

Corollary 5.10. *Subspace of totally bounded metric space is totally bounded.*

Theorem 5.11. *Metric M totally bounded iff every sequence in M has Cauchy subsequence.*

Proof. ($\Rightarrow$) Let $x_n \in M$. M covered by finitely many balls radius $\frac{1}{2}$ so $\exists\, B_1$ s.t. $N_1 = \{n \in \mathbb{N} : x_n \in B_1\}$ has $|N_1| = \infty$.

Suppose inductively have defined infinite $N_{k-1} \subset \mathbb{N}$. Since M covered by finitely many balls of radius $\frac{1}{2k}$ $\exists$ one ball B_k s.t. $N_k = \{n \in N_{k-1} : x_n \in B_k\}$ is infinite.

Let $n(1)$ be least element of N_1, $n(k)$ least element of N_k s.t. $n(k) > n(k-1)$. Then $\left(x_{n(k)}\right)_{k=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$ s.t. $\forall\, k \quad x_{n(i)} \in B_k$ for $i \geq k$. Hence $d\left(x_{n(i)}, x_{n(j)}\right) < \frac{1}{k} \forall\, i, j \geq k$ so $\left(x_{n(k)}\right)$ Cauchy.

($\Leftarrow$) Suppose M not totally bounded. Then for some $\varepsilon > 0$ $\nexists$ finite F with all points of M within ε of it. Choose $x_1 \in M$, inductively x_k s.t. $d(x_k, x_i) \geq \varepsilon \quad \forall\, i < k$. x_k exists by assumption M not totally bounded.

This gives sequence $(x_k)_{k=1}^{\infty}$ s.t. $d(x_i, x_j) \geq \varepsilon \quad \forall\, i \neq j$. Then no subsequence of (x_k) Cauchy. ☐

5.6 Completeness and Compactness

Theorem 5.12. *Subspace C of complete metric M compact iff closed and totally bounded.*

Proof. ($\Rightarrow$) C closed by 3.3, totally bounded since $\forall\, \varepsilon > 0$ open cover $B(x, \varepsilon)$ $(x \in C)$ has finite subcover.

($\Leftarrow$) Every sequence in C has Cauchy subsequence by 5.11, converges to point of M since M complete. C closed so limit in C. ☐

Lemma 5.13. *If M subspace of N totally bounded so is $\overline{M}$.*

Proof. Fix $\varepsilon > 0$. Let $F \subset M$ be finite s.t. $M \subset \bigcup_{x \in F} B\left(x, \frac{\varepsilon}{2}\right)$. Then

$$\overline{M} \subset \bigcup_{x \in F} \overline{B\left(x, \frac{\varepsilon}{2}\right)} \subset B\left(x, \varepsilon\right).$$

☐

Theorem 5.14. *Subspace S of complete metric M totally bounded iff $\overline{S}$ compact.*

Proof. ($\Rightarrow$) $\overline{S}$ totally bounded by 5.13 so compact by 5.12.

($\Leftarrow$) $\overline{S}$ totally bounded by 5.12 and so is $S \subset \overline{S}$ by 5.10. ☐

5.7 Cantor's theorem

Definition 5.6. *Diameter* of $\emptyset \neq S \subset M$ defined by

$$\operatorname{diam}(S) = \sup_{x, y \in S} d(x, y)$$

Theorem 5.15 (Cantor)**.** *Let F_n decreasing sequence of non-empty closed subsets of metric M s.t.* $\operatorname{diam}(F_n) \xrightarrow[n \to \infty]{} 0$. *Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

Proof. Pick $x_n \in F_n$. Then $\forall\, i \geq n$, $x_i \in F_i \subset F_n$.

Hence for $i, j \geq n$, $d(x_i, x_j) \leq \operatorname{diam}(F_n)$. Hence (x_n) Cauchy. Converges to some x as M complete.

F_n closed so $x \in F_n$. Hence $x \in \bigcap_{n=1}^{\infty} F_n$. ☐

5.8 Baire category theorem

Definition 5.7. $S \subset M$ is

- *dense* in M if $\overline{S} = M$.
- *nowhere dense* in M if $M \setminus \overline{S}$ is dense in M.
- *meagre* in M if it is the union of a sequence of nowhere dense sets.

Proposition 5.16. *$S \subset M$ nowhere dense in M iff $\overline{S}^\circ = \emptyset$*

Proof. $\overline{S}^\circ = \emptyset = M \setminus \overline{(M \setminus \overline{S})}$ so if RHS $= \emptyset$ then $M \setminus \overline{S}$ is dense in M so S is nowhere dense. Conversely if S is nowhere dense in M then $M \setminus \overline{S} = M$ so RHS $= \emptyset$. ☐

Theorem 5.17 (Baire Category)**.** *A complete metric space is not meagre in itself.*

I.e. if S_n are the nowhere dense subsets of non-empty complete M then

$$M \setminus \bigcup_{n=1}^{\infty} S_n \neq \emptyset$$

Proof. IDEA: FIND DECREASING SEQUENCE OF DENSE SETS WITH NON-EMPTY INTERSECTION OF THEIR CLOSURES BY CANTOR. ANY POINT IN THIS INTERSECTION CANNOT BE IN ANY NOWHERE DENSE SET.

$G_k := M \setminus \overline{S_k}$ dense in M, open.

Then $G_1 \neq \emptyset$. Choose $x_1 \in G_1$ and $\delta_1 > 0$ s.t. $B(x_1, \delta_1) \subset G_1$.

Continue inductively: Having defined x_{k-1}, δ_{k-1} use fact that G_k dense to find $x_k \in G_k \cap B\left(x_{k-1}, \frac{\delta_{k-1}}{2}\right)$. Find $0 < \delta_k < \frac{\delta_{k-1}}{2}$ s.t. $B(x_k, \delta_k) \subset G_k$.

$\delta_k \underset{k\to\infty}{\longrightarrow} 0$ and $\forall\, k$, $\overline{B(x_k, \delta_k)} \subset B(x_{k-1}, \delta_{k-1})$.

Then by Cantor (5.15) $\bigcap_{k=1}^{\infty} \overline{B(x_k, \delta_k)} \neq \emptyset$. Let x be in this intersection. Then $x \in B(x_k, \delta_k) \subset G_k \quad \forall\, k$ so $x \notin S_k \quad \forall\, k$. Hence these is a point x that is not in the union of all nowhere dense subsets of M, so M cannot be meagre. ☐

Proposition 5.18. *The Cantor set $\mathfrak{C}$ is uncountable.*

Proof. $\forall\, x \in \mathfrak{C}$ there are points $y \in \mathfrak{C}$, $y \neq x$ arbitrarily close to x. In other words, $\mathfrak{C} \setminus \{x\}$ is dense in $\mathfrak{C}$. Therefore $\{x\}$ is nowhere dense in $\mathfrak{C}$ as it is closed.

If $\mathfrak{C}$ were countable would have $\mathfrak{C} = \bigcup_{j=1}^{\infty} \{x_j\}$ showing $\mathfrak{C}$ meagre in itself. This contradicts Baire's theorem (5.17). ☐

Lemma 5.19. *Let $f\colon [1, \infty) \to \mathbb{R}$ be cts s.t. for some $a \in \mathbb{R}$ $\exists$ arbitrarily large x with $f(x) < a$. Then $\forall\, k \in \mathbb{N} : S = \bigcup_{n=k}^{\infty} \{x \in [1, \infty) : f(nx) \geq a\}$ is nowhere dense.*

Proof. f cts so S closed. Let $1 \le \alpha < \beta < \infty$. RTP $(\alpha, \beta) \setminus S \ne \emptyset$.

For large n, $\frac{n+1}{n} < \frac{\beta}{\alpha}$ so $(n+1)\alpha < n\beta$. Then $\bigcup_{n=k}^{\infty}(n\alpha, n\beta)$ contains some (r, ∞) and so a point y s.t. $f(y) < a$.

Find n s.t. $y \in (n\alpha, n\beta)$. Then $x = \frac{y}{n} \in (\alpha, \beta)$ and $f(nx) < a$ so $x \notin S$. ☐

Proposition 5.20. *Let* $f\colon [1, \infty) \to \mathbb{R}$ *be cts s.t.* $\forall x \ge 1$*,* $\lim_{n\to\infty} f(nx)$ *exists. Then* $\lim_{x\to\infty} f(x)$ *exists.*

Proof. If $\lim_{x\to\infty} f(x)$ not exist then $\exists\, a, b;\ a < b$ s.t. $\exists$ arbitrarily large x, y with $f(x) < a$, $f(y) > b$.

Then by previous lemma:

$$T = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in [1, \infty) : f(nx) \ge a\} \cup \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in [1, \infty) : f(nx) \le b\}$$

is meagre. By Baire (5.17) $\exists\, x \notin T$.

x not in first union so $\forall\, k \quad \exists\, n \ge k$ s.t. $f(nx) < a$. x not in second union so $\forall\, k \quad \exists\, m \ge k$ s.t. $f(mx) > b$. Hence $f(nx)$ not converge. ☐

Theorem 5.21. $\exists\, f \in C[0, 1]$ *not differentiable at any point.*

Proof. IDEA: $C[0, 1]$ IS COMPLETE. FUNCTIONS WITH DERIVATIVE AT AT LEAST ONE POINT FORM A MEAGRE SUBSET. RESULT BY BAIRE.

Define S_n:

$$S_n = \{f \in C[0, 1] : (\exists\, x \in [0, 1])(\forall\, y \in [0, 1])\ |f(y) - f(x)| \le n|y - x|\}$$

Summary:

1. S_n closed.
2. S_n nowhere dense as has dense complement and closed.
3. If $f'(x)$ exists for some x then $f \in S_n$ for some n.

1. Let $f_k \in S_n$, $f_k \to f$. Find $x_k \in [0, 1]$ s.t. $\forall\, y \in [0, 1]$,

   $$|f_k(y) - f_k(x_k)| \le n|y - x_k|$$

   x_k has convergent subsequence so assume $x_k \to x$. By uniform convergence

   $$|f(y) - f(x)| \le n|y - x|$$

   Therefore $f \in S_n$, so S_n closed.

2. Let $g \in C[0,1]$, $\varepsilon > 0$. g uniformly cts so $\exists\, \delta > 0$ s.t.

$$|x-y| \le \delta \Rightarrow |g(x)-g(y)| < \frac{\varepsilon}{4} \tag{1}$$

Let $x_i = \frac{i}{k}$, $\varphi(x) = k\varepsilon \min_{0\le i\le k} |x-x_i|$. Then $0 \le \varphi \le \frac{\varepsilon}{2}$ and suffices to show $f = \varphi + g \notin S_n$.

Suppose $f \in S_n$ and find x "responsible for it".

Choose $1 \le j \le k$ s.t. $x \in [x_{j-1}, x_j]$. Let $y = \frac{x_{j-1}+x_j}{2}$. Then

$$\begin{aligned}\frac{\varepsilon}{2} &= |\varphi(y) - \varphi(x_j)| \\ &\le |f(y) - f(x_j)| + |g(y) - g(x_j)| \\ &\underset{(1)}{\le} |f(y) - f(x)| + |f(x_j) - f(x)| + \frac{\varepsilon}{4} \\ &\le n|y-x| + n|x_j - x| + \frac{\varepsilon}{4} \\ &\le \frac{2n}{k} + \frac{\varepsilon}{4} \\ &< \frac{\varepsilon}{2}\end{aligned}$$

This is a contradiction. So $f \notin S_n$.

3. If $f'(x)$ exists find $\delta > 0$ s.t. $\forall\, 0 < |y-x| < \delta$,

$$\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| < 1 \Longrightarrow \left|\frac{f(y)-f(x)}{y-x}\right| < 1 + |f'(x)|$$

Function $y \mapsto \frac{f(y)-f(x)}{y-x}$ is continuous on $[0,1] \setminus (x-\delta, x+\delta)$ which is compact. Hence the function is bounded, so $\exists\, n \in \mathbb{N}$ s.t.

$$y \in [0,1] \setminus (x-\delta, x+\delta) \Longrightarrow \left|\frac{f(y)-f(x)}{y-x}\right| \le n$$

May take $n > 1 + |f'(x)|$ so get inequality holding $\forall\, y \in [0,1] \setminus \{x\}$.

Then $|f(y) - f(x)| \le n|y-x| \quad \forall\, y \in [0,1]$. (This clearly holds for $y = x$ and holds by the above for $y \ne x$.) So if $\exists\, f \in C[0,1]$ s.t. $f'(x)$ exists for some x then $f \in S_n$.

These three parts together complete the proof, since by Baire (5.17) $C[0,1]$ is not meagre, so there must be a function which is not differentiable at any point, as any that are differentiable at at least one point are in a nowhere dense subset. □

5.9 Compactness and Cantor set

Theorem 5.22. *Every compact metric M is continuous image of Cantor set $\mathfrak{C}$.*

Proof. Let $A_k \subset M$ be finite s.t. $\forall\, x \in M \quad d(A_k, x) \leq 2^{-k}$.

By induction construct sequence of cts functions $f_k \colon \mathfrak{C} \to M$ s.t. $f_k(\mathfrak{C}) = A_k$, $d(f_k(x), f_{k+1}(x)) \leq 2^k \quad \forall\, x \in C$.

(f_k) Cauchy in $C(\mathfrak{C}, M)$ so converge to cts $f \colon \mathfrak{C} \to M$. $f(\mathfrak{C})$ dense in M. Also compact, so closed, hence $f(\mathfrak{C}) = M$. ☐

Corollary 5.23. $\exists$ *continuous surjective map* $f \colon [0,1] \to [0,1]^2$.

Proof. Extend surjective cts $f \colon \mathfrak{C} \to [0,1]^2$ linearly to each interval removed during construction of $\mathfrak{C}$. ☐