

MA250

Introduction to Partial Differential Equations

Ian Rogers

Based on the lectures of Dr Florian Theil

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0 Notation and concepts

$$u = u(x, y, \dots) \quad u_x := \frac{\partial u}{\partial x} \quad u_{xx} := \frac{\partial^2 u}{\partial x^2}$$

$$F(x, y, u, u_x, u_y, \dots) = 0 \tag{0.i}$$

(0.i) is called an *ODE* if only has derivatives w.r.t. one variable. If F depends only on x, y, u_x, u_y, \dots but not higher order derivatives (0.i) is called *1st order*. If F depends linearly on u, u_x, u_y, \dots then (0.i) is *linear*.

1 First order linear equations

1.1 Transport equation

$$u_t = -cu_x \tag{1.i}$$

Heterogeneous case $c = c(x, t)$. Interested in $M(t) = \int_a^b u(x, t) dx$.

Compute change in M by flux through boundary of $[a, b]$.

$$j(t) = c(a, t)u(a, t) - c(b, t)u(b, t)$$

Let $a = x, b = x + h$.

$$\begin{aligned} \frac{d}{dt} \int_x^{x+h} u(s, t) ds &= c(x, t)u(x, t) - c(x+h, t)u(x+h, t) \\ &\implies \\ \frac{1}{h} \int_x^{x+h} u_t(s, t) ds &= -\frac{c(x+h, t)u(x+h, t) - c(x, t)u(x, t)}{h} \\ &\xrightarrow{h \rightarrow 0} \\ &u_t = -(cu)_x \\ &\iff \\ u_t + cu_x + c_x u &= 0 \end{aligned}$$

1.2 Method of characteristics

$$au_x + bu_y = 0 \tag{1.ii}$$

$(a, b) \neq \mathbf{0}, a, b$ constant.

1.2.1 Geometric method

$au_x + bu_y$ directional derivative of u at (x, y) in direction $\mathbf{v} = (a, b)$. Then (1.ii) $\implies u$ constant in direction \mathbf{v} .

Assume w.l.o.g that $b \neq 0$. $(x, y) = (x_0 + ta, tb)$ where $t = \frac{y}{b}$, $x_0 = x - y\frac{a}{b}$. This corresponds to moving along characteristics to where $y = 0$. Then

$$u(x, y) = f(x_0) = f\left(x - y\frac{a}{b}\right)$$

is a complete solution of (1.ii).

1.2.2 Co-ordinate method

$$\begin{aligned}x' &= ax + by \\y' &= bx - ay \\u_x &= \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial x} \\&= u_{x'}a + u_{y'}b \\u_y &= \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial y} \\&= u_{x'}b - u_{y'}a\end{aligned}$$

so $au_x + bu_y = 0 \iff a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = 0 \iff (a^2 + b^2)u_{x'} = 0$
so we now can solve this ODE.

$$\begin{aligned}u(x', y') &= u(0, bx - ay) \\&= g(bx - ay) \\&= f\left(x - y\frac{a}{b}\right)\end{aligned}$$

where $g(t) = f(bt)$.

1.3 Insights

1. There is no hope to find a general solution for all first order linear PDEs, since not all first order ODEs have a solution formula.
2. Can solve (1.ii) by integrating the ODE $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$.
3. Solutions of PDEs not necessarily smooth.

2 Wave equation

$$u_{tt} = c^2 u_{xx} \tag{2.i}$$

$$\rho u_{tt} = T u_{xx} \tag{2.ii}$$

c is the wave speed, $u(x, t)$ is the vertical displacement of vibrating string. In alternative formulation (2.ii) ρ is density of string, T tension.

2.1 Construction of solutions on real line

Simplest possible unbounded setting is $u = u(x, t)$, $x \in \mathbb{R}$, $t \geq 0$.

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u$$

This means we can compute $v = u_t + cu_x$. Then $v_t - cv_x = 0$.

Introduce new co-ordinates $\xi = x + ct$, $\eta = x - ct$. Then

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial t} &= c \frac{\partial}{\partial \xi} - c \frac{\partial}{\partial \eta} \end{aligned}$$

Then $\left(-2c \frac{\partial}{\partial \xi}\right) \left(2c \frac{\partial}{\partial \eta}\right) u = 0$ so (2.i) becomes

$$u_{\eta\xi} = 0 \tag{2.iii}$$

since $c \neq 0$. The solution of (2.iii) is given by $u(\xi, \eta) = f(\xi) + g(\eta)$, so

$$u(x, t) = f(x + ct) + g(x - ct) \tag{2.iv}$$

2.2 Initial value problem

$$\begin{cases} u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases} \tag{2.v}$$

Solution of (2.i), (2.v) is easily found using (2.iv).

$$\begin{aligned} f + g &= \phi \\ cf' - cg' &= \psi \end{aligned}$$

Then $\phi' = f' + g'$, $\frac{1}{c}\psi = f' - g'$.

$$\begin{aligned} f' &= \frac{1}{2} \left(\phi' + \frac{1}{c}\psi \right) \\ g' &= \frac{1}{2} \left(\phi' - \frac{1}{c}\psi \right) \\ \implies \\ f(s) &= \frac{1}{2}\phi(s) + \frac{1}{2c} \int_0^s \phi(r) dr + A \\ g(s) &= \frac{1}{2}\phi(s) - \frac{1}{2c} \int_0^s \phi(r) dr + B \end{aligned}$$

$f + g = \phi$ so $A + B = 0$ and

$$\begin{aligned} u(x, t) &= \frac{1}{2}\phi(x + ct) + \frac{1}{2}\phi(x - ct) + \frac{1}{2c} \int_0^{x+ct} \phi(r) dr - \frac{1}{2c} \int_0^{x-ct} \phi(r) dr \\ &= \frac{1}{2} \left(\phi(x + ct) + \phi(x - ct) + \frac{1}{c} \int_{x-ct}^{x+ct} \psi(r) dr \right) \end{aligned} \tag{2.vi}$$

(2.vi) is called "*D'Alembert's Solution*".

2.3 Causality and energy conservation

2.3.1 Causality principle

Value of initial position ϕ and initial velocity ψ at x_0 affects solution $u(x, t)$ only for $x \in [x_0 - ct, x_0 + ct]$.

Hence if $\psi(x) = \phi(x) = 0$ for $|x| > R$ then $u(x, t) = 0 \quad \forall |x| > R + ct$.

2.3.2 Energy conservation

Define $E(t) = K(t) + P(t)$ where

$$\begin{aligned} K(t) &= \frac{\rho}{2} \int_{-R}^R u_t^2(x, t) dx && \text{kinetic energy} \\ P(t) &= \frac{T}{2} \int_{-R}^R u_x^2(x, t) dx && \text{potential energy} \end{aligned}$$

Theorem 1 (Energy conservation). *Let (2.ii) have the properties*

1. $u \in C^2(\mathbb{R} \times \mathbb{R})$
2. $u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$
3. $\psi(x) = \phi(x) = 0$ if $|x| > R$.

Then $E(t) = E(0) = \frac{1}{2} \int_{-\infty}^{\infty} (\rho|\psi(x)|^2 + T|\phi_x(x)|^2) dx$.

Proof. $S := R + ct$. Then $\forall S \in [0, t]$

$$\begin{aligned} E(s) &= \frac{1}{2} \int_{-R}^R (\rho u_t^2(x, s) + T u_x^2(x, s)) dx \\ \frac{dE}{ds}(s) &= \int_{-R}^R (\rho u_t u_{tt} + T u_x u_{xt}) dx \\ &\stackrel{[\rho u_{tt} = T u_{xx}]}{=} T \int_{-R}^R (u_t u_{xx} + u_x u_{xt}) dx \\ &\stackrel{[\text{parts}]}{=} T \int_{-R}^R \underbrace{(u_t u_{xx} - u_x u_{xt})}_{=0} dx + T \underbrace{(u_t u_x(R, S) - u_t u_x(-R, S))}_{=0 \text{ since } u(\pm R, S) = 0 \forall S \in [0, t]} \end{aligned}$$

□

3 Diffusion equation

$$u_t = ku_{xx} \quad k > 0 \quad (3.i)$$

Theorem 2 (Maximum principle). *Let $u \in C^2([0, l] \times [0, T])$ be solution of (3.i). Then u assumes its maximum on the set $\{(x, t) \in [0, l] \times [0, T] : t = 0 \text{ or } x = 0 \text{ or } x = l\}$.*

Initial and boundary conditions:

$$\begin{cases} u(x, 0) = \phi(x) \\ u(0, t) = g(t) \\ u(l, t) = h(t) \end{cases} \quad (3.ii)$$

Theorem 3 (Uniqueness). *Let $u \in C^2([0, l] \times [0, T])$ be solution of (3.i) and satisfy IBC (3.ii). Then u is unique.*

Proof. Let $u_1, u_2 \in C^2([0, l] \times [0, T])$ be two solutions (3.i) and satisfy IBC (3.ii) with the same functions ϕ, g, h . Define $v(x, t) = u_1(x, t) - u_2(x, t)$.

(3.i) is linear so v satisfies (3.i), and $v(x, 0) = v(0, t) = v(l, t) = 0$. By the maximum principle, and the analogous minimum principle $v \equiv 0$. \square

Theorem 4 (Stability). *Let $u_1, u_2 \in C^2([0, l] \times [0, T])$ be two solutions of inhomogeneous $u_t = ku_{xx} + f(x, t)$. If u_1, u_2 satisfy $u_1(x, 0) = \phi_1(x)$, $u_2(x, 0) = \phi_2(x)$ and $u_1(0, t) = u_2(0, t) = g(t)$ and $u_1(l, t) = u_2(l, t) = h(t)$ then*

$$\int_0^l (u_1(x, t) - u_2(x, t))^2 dx \leq \int_0^l (\phi_1(x) - \phi_2(x))^2 dx \quad (3.iii)$$

Proof. $v(x, t) := u_1(x, t) - u_2(x, t)$. Then

$$\begin{cases} v_t = kv_{xx} \\ v(x, 0) = \phi_1(x) - \phi_2(x) \\ v(0, t) = v(l, t) = 0 \end{cases} \quad (3.iv)$$

$$\begin{aligned} \frac{d}{dt} \int_0^l v^2(x, t) dx &= 2 \int_0^l v(x, t) v_t(x, t) dx \\ &= 2 \int_0^l v(x, t) kv_{xx}(x, t) dx \\ &\stackrel{[\text{parts}]}{=} -2k \int_0^l v_x^2(x, t) dx + \underbrace{v(0, t)v_x(0, t) - v(l, t)v_x(l, t)}_{= 0 \text{ by (3.iv)}} \\ &= -2k \int_0^l v_x^2(x, t) dx \leq 0 \end{aligned}$$

so

$$\begin{aligned} \int_0^l v^2(x, t) dx &\leq \int_0^l v^2(x, 0) dx \\ &= \int_0^l (\phi_1(x) - \phi_2(x))^2 dx \quad \square \end{aligned}$$

Remark 1. Inequalities like (3.iii) are often called *energy estimates*.

3.1 Existence of solutions of (3.i) on the real line

We want $u \in C^2(\mathbb{R} \times [0, \infty))$.

Note that if u solves (3.i) then:

- (a) translates $u((x - y), t)$
- (b) derivatives u_x, u_t, \dots
- (c) linear combinations $c_1 u_1 + c_2 u_2$
- (d) integrals e.g. $\int_0^x u(s, t) ds$
- (e) dilated functions $f_a(x, t) = u(\sqrt{a}x, at), \quad a > 0$

solve (3.i) as well.

Step 1 Look for solutions of the form $Q(x, t) = g(p)$ where $p = \frac{x}{\sqrt{4kt}}$. Then dilates are solutions.

Step 2 Plug $g(p)$ into (3.i), get ODE:

$$0 = Q_t - kQ_{xx} = -\frac{1}{t} \left(\frac{1}{2} p g'(p) + \frac{1}{4} g''(p) \right)$$

so $g'' + 2pg' = 0$. Hence $g(p) = c_1 \int_0^p e^{-s^2} ds + c_2$.

Step 3 Consider behaviour of Q as $t \rightarrow 0$ to determine c_1, c_2 .

If $x > 0$

$$\begin{aligned} \lim_{t \downarrow 0} Q(x, t) &= \lim_{t \downarrow 0} c_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds + c_2 \\ &= c_1 \int_0^\infty e^{-s^2} ds + c_2 \\ &= c_1 \frac{\sqrt{\pi}}{2} + c_2 \end{aligned}$$

If $x < 0$ $\lim_{t \downarrow 0} Q(x, t) = -c_1 \frac{\sqrt{\pi}}{2} + c_2$.

Pick $c_1 = \frac{1}{\sqrt{\pi}}, c_2 = \frac{1}{2}$. Then

$$\lim_{t \downarrow 0} Q(x, t) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \quad (3.v)$$

Step 4 Define

$$S(x, t) = \frac{\partial Q}{\partial x}(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

is also a solution of (3.i) by (b) above. This is called a *fundamental solution*.

$$\int_{-\infty}^{\infty} S(x, t) dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp = 1$$

This makes sense as this models the diffusion of a substance.

Step 5 Given ϕ define

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \phi(y) dy$$

Step 6 Analyse $u(x, t)$ as $t \rightarrow 0$. Would expect $u(x, t) \xrightarrow{t \rightarrow 0} \phi(x)$.

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{\infty} Q_x(x - y, t) \phi(y) dy \\ &= - \int_{-\infty}^{\infty} Q_y(x - y, t) \phi(y) dy \\ &= \int_{-\infty}^{\infty} Q(x - y, t) \phi'(y) dy - Q(x - y, t) \phi(y) \Big|_{y=-\infty}^{\infty} \end{aligned}$$

Assuming $\lim_{|x| \rightarrow \infty} |\phi(x)| = 0$ drop 2nd term.

Then

$$\begin{aligned} \lim_{t \downarrow 0} u(x, t) &= \lim_{t \downarrow 0} \int_{-\infty}^{\infty} Q(x - y, t) \phi'(y) dy \\ &= \int_{-\infty}^x \phi'(y) dy \\ &\text{by (3.v)} \\ &= \phi(x) - \phi(-\infty) = \phi(x) \end{aligned}$$

3.2 Comparison between (2.i) and (3.i)

Property	Diffusion equation (3.i)	Wave equation (2.i)
Speed of propagation	infinite	finite ($\leq c$)
Singularities at $t > 0$	immediately destroyed	travel along characteristics
Solutions exist for $t > 0$	✓	✓
Solutions exist for $t < 0$	✗	✓
Maximum principle	✓	✗
"Information"	Lost gradually since can't solve for t_1 if know about $t_2 > t_1$.	Transported

4 Introduction to Fourier Analysis

4.1 Boundary Values

Dirichlet boundary conditions $u(0, t) = a, u(l, t) = b$. Corresponds to string clamped at both ends, known temperature at boundary of container.

Neumann boundary conditions $u_x(0, t) = a, u_x(l, t) = b$. Corresponds to pulling string with constant force at ends, knowing flux through the boundary.

4.2 Separation of variables

Consider (2.i) with Dirichlet bcs $u(0, t) = u(l, t) = 0$. Fix special initial conditions $u(x, 0) = a_k \sin\left(\frac{k\pi x}{l}\right), u_t(x, 0) = b_k \sin\left(\frac{k\pi x}{l}\right)$.

Construct solutions assuming $u(x, t) = v(t) \sin\left(\frac{k\pi x}{l}\right)$ and plug this into (2.i):

$$v''(t) \sin\left(\frac{k\pi x}{l}\right) = -\left(c^2 v(t) \frac{k^2 \pi^2}{l^2} \sin\left(\frac{k\pi x}{l}\right)\right).$$

If we can solve $v'' = -c^2 \frac{k^2 \pi^2}{l^2} v$ then u solves the wave equation.

$$v(t) = \alpha_k \cos\left(\frac{ck\pi}{l} t\right) + \beta_k \sin\left(\frac{ck\pi}{l} t\right)$$

Initial conditions give $\alpha_k = a_k, \beta_k = \frac{l}{ck\pi} b_k$.

By linearity

$$u(x, t) = \sum_{k=1}^n \left[\left(a_k \cos\left(\frac{ck\pi}{l} t\right) + \frac{b_k l}{ck\pi} \sin\left(\frac{ck\pi}{l} t\right) \right) \sin\left(\frac{k\pi x}{l}\right) \right] \quad (4.i)$$

solves the IBVP

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0, t) = u(l, t) = 0 \\ u(x, 0) = \phi(x) = \sum_{k=1}^n a_k \sin\left(\frac{k\pi x}{l}\right) \\ u_t(x, 0) = \psi(x) = \sum_{k=1}^n b_k \sin\left(\frac{k\pi x}{l}\right) \end{cases} \quad (4.ii)$$

4.3 Fourier coefficients

Definition 1. $f: \mathbb{R} \rightarrow \mathbb{C}, k \in \mathbb{Z}$. $f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$. $\hat{f}(k) \in \mathbb{C}$ are the *Fourier coefficients*.

$f(x) \in \mathbb{R}$ if $\hat{f}(-k) = \overline{\hat{f}(k)}$.

Definition 2. $\forall n \in \mathbb{N}$ the n th partial Fourier series given by $f_n(x) = \sum_{k=-n}^n \hat{f}(k) e^{ikx}$.

f_n is 2π -periodic, so it can only converge to a 2π -periodic function.

4.3.1 Calculating $\hat{f}(k)$

$$\int_0^{2\pi} e^{ikx} e^{ik'x} dx = \begin{cases} 2\pi & \text{if } k' = -k \\ 0 & \text{if } k' \neq -k \end{cases} \quad (4.iii)$$

Assume $f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$. Let $k' \in \mathbb{Z}$. $e^{ik'x} f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{i(k+k')x}$, so

$$\begin{aligned} \int_0^{2\pi} e^{ik'x} f(x) dx &= \sum_{k \in \mathbb{Z}} \hat{f}(k) \int_0^{2\pi} e^{i(k+k')x} dx \\ &\stackrel{\text{by (4.iii)}}{\iff} \\ \int_0^{2\pi} f(x) e^{-ikx} dx &= 2\pi \hat{f}(k) \\ &\iff \\ \hat{f}(k) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx \end{aligned}$$

Theorem 5. Let $f \in C^0([0, 2\pi], \mathbb{C})$. Among all choices of $2n + 1$ constants c_{-n}, \dots, c_n the choice that minimises

$$E_n = \int_0^{2\pi} \left| f(x) - \sum_{k=-n}^n 2^{ikx} c_k \right|^2 dx$$

is $c_k = \hat{f}(k)$.

Remark 2. c_k not depend on n .

4.3.2 Consequences

1. BESSEL'S INEQUALITY:

$$2\pi \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 \leq \int_0^{2\pi} |f(x)|^2 dx$$

2. PARSEVAL'S EQUALITY:

If $\int_0^{2\pi} \left| f(x) - \sum_{k \in \mathbb{Z}} s^{ikx} \hat{f}(k) \right|^2 dx = 0$ then

$$\int_0^{2\pi} |f(x)|^2 dx = 2\pi \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2$$

3. RIEMANN-LEBESQUE LEMMA

Lemma 6 (Riemann-Lebesque).

$$\lim_{k \rightarrow \infty} \int_0^{2\pi} \sin(kx) f(x) dx = \lim_{k \rightarrow \infty} \int_0^{2\pi} \cos(kx) f(x) dx = 0 \quad \forall f \in C^0([0, 2\pi], \mathbb{C})$$

Proof. Assume f is real-valued. If not prove for real and imaginary parts separately.

$$\left| \int_0^{2\pi} \sin(kx) f(x) dx \right| = \left| 2\pi \operatorname{Im} \left(\hat{f}(k) \right) \right| \leq 2\pi \left| \hat{f}(k) \right|$$

Since $\sum_{k \in \mathbb{Z}} \left| \hat{f}(k) \right|^2 < \infty$, $\lim_{k \rightarrow \infty} \left| \hat{f}(k) \right|^2 = 0$ □

Definition 3 (3 notions of convergence). Let $f, f_n: [0, 2\pi] \rightarrow \mathbb{C}$. $f_n \rightarrow f$ in the:

Pointwise sense if $\lim_{n \rightarrow \infty} f_n(x) = f(x) \forall x \in [0, 2\pi]$.

Uniform sense if $\lim_{n \rightarrow \infty} \sup_{x \in [0, 2\pi]} |f_n(x) - f(x)| = 0$.

Mean-square (or L^2) sense if $\lim_{n \rightarrow \infty} \int_0^{2\pi} |f_n(x) - f(x)|^2 dx = 0$.

Remark 3. $\int_0^{2\pi} |f_n(x) - f(x)|^2 dx \leq 2\pi \sup_{x \in [0, 2\pi]} |f_n(x) - f(x)|^2$ so uniform convergence $\Rightarrow L^2$ convergence. Also uniform \Rightarrow pointwise. No other implications true.

Theorem 7 (Pointwise convergence of Fourier series). Let $f \in C^1(\mathbb{R})$ be 2π -periodic. Then

$$\lim_{n \rightarrow \infty} \sum_{|k| \leq n} e^{ikx} \hat{f}(k) = f(x) \quad \forall x \in [0, 2\pi].$$

Theorem 8. Let f be 2π -periodic piecewise C^1 function (i.e. $\exists D \subset [0, 2\pi]$ s.t. $f \in C^1(\mathbb{R} \setminus (D + 2\pi\mathbb{Z}))$, $|D| < \infty$).

If $\forall x_0 \in D$, $\lim_{x \downarrow x_0} f(x)$ and $\lim_{x \uparrow x_0} f(x)$ exist then $f_n(x) = \sum_{k=-n}^n e^{ikx} \hat{f}(k)$ converges pointwise as $n \rightarrow \infty$ to

$$\begin{cases} f(x) & \forall x \in \mathbb{R} \setminus (D + 2\pi\mathbb{Z}) \\ \frac{1}{2} \lim_{x \uparrow x_0} f(x) + \frac{1}{2} \lim_{x \downarrow x_0} f(x) & \forall x \in (D + 2\pi\mathbb{Z}) \end{cases}$$

Theorem 9. 1. $f \in C^s(\mathbb{R})$, 2π -periodic. Then $\exists c > 0$ s.t. $\left| \hat{f}(k) \right| \leq \frac{c}{|k|^s}$.
This means decay rate of Fourier coefficients controlled by smoothness of f .

2. Let $\hat{f}(k) \in \mathbb{C}$, $k \in \mathbb{Z}$ s.t. $\left| \hat{f}(k) \right| \leq \frac{c}{|k|^r}$ for some $c > 0$, $r > 1$, $r \in \mathbb{R}$. Then

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \sum_{k=-n}^n e^{ikx} \hat{f}(k)$$

exists and $f \in C^s(\mathbb{R})$, for $s \in (\mathbb{N} \cup \{0\}) \cap [0, r - 1)$.

Proof. 1.

$$\begin{aligned}
|\hat{f}(k)| &= \frac{1}{2\pi} \left| \int_0^{2\pi} f(x) e^{-ikx} dx \right| \\
&\stackrel{[\text{parts}]}{=} \frac{1}{2\pi} \left| \frac{1}{ik} \int_0^{2\pi} f'(x) e^{ikx} dx \right| \\
&= \dots \\
&= \frac{1}{2\pi} \frac{1}{|k|^s} \left| \int_0^{2\pi} f^{(s)}(x) e^{-ikx} dx \right| \\
&\leq \frac{1}{2\pi} \frac{2\pi}{|k|^s} \sup_{x \in [0, 2\pi]} |f^{(s)}(x)|
\end{aligned}$$

Let $c = \sup_{x \in [0, 2\pi]} |f^{(s)}(x)|$, so $|\hat{f}(k)| \leq c|k|^{-s}$ for $k \in \mathbb{Z} \setminus \{0\}$.

2. By **Analysis III** if $f_n \in C^s(\mathbb{R})$ sequence of functions converging uniformly to f and sth derivative converges uniformly then $f \in C^s(\mathbb{R})$.

$$\begin{aligned}
f_n(x) &= \sum_{k=-n}^n e^{ikx} \hat{f}(k) \\
f_n^{(s)}(x) &= \sum_{k=-n}^n (ik)^s e^{ikx} \hat{f}(k)
\end{aligned}$$

$f_n^{(s)}$ converge uniformly if $|k|^s |\hat{f}(k)| \leq \frac{c}{|k|^{1+\varepsilon}}$ for some $\varepsilon > 0$. True if $s < r - 1$. \square

Theorem 10. Let $f \in C^2(\mathbb{R})$ be 2π -periodic. Then

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |f(x) - f_n(x)| = 0$$

Proof. $f \in C^2(\mathbb{R})$ so $s = 2$.

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq \sum_{|k| > \min\{m, n\}} |e^{ikx} \hat{f}(k)| \\
&\leq \sum_{|k| > \min\{m, n\}} |\hat{f}(k)| \\
&\stackrel{\text{by thm 9}}{\leq} 2 \sup_{x \in [0, 2\pi]} |f''(x)| \sum_{k=\min\{m, n\}+1}^{\infty} \frac{1}{k^2} \\
&\leq 2 \sup_{x \in [0, 2\pi]} |f''(x)| \int_{k=\min\{m, n\}+1}^{\infty} \frac{1}{(r-1)^2} dr \\
&= \frac{2}{\min\{m, n\}} \sup_{x \in [0, 2\pi]} |f''(x)|.
\end{aligned}$$

If we choose $\min\{m, n\} > \frac{2 \sup_{x \in [0, 2\pi]} |f''(x)|}{\varepsilon}$ then $|f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in \mathbb{R}$. This does not depend on x so $f_n(x)$ converges uniformly. \square

4.4 Gibbs phenomenon

When f is piecewise C^1 we need to investigate the convergence of the Fourier series near the discontinuities. It converges to the average of the limits from either side. At the discontinuities the Fourier series overshoots by about 9%. This is called the Gibbs phenomenon.

4.5 Back to PDEs

Here we construct 2π -periodic solutions of the wave equation. Use separation Ansatz

$$u(x, t) = \sum_{k \in \mathbb{Z}} \hat{u}(k, t) e^{ikx}$$

Recall (2.i) $u_{tt} = c^2 u_{xx}$. Here we use initial conditions:

$$\begin{aligned} u(x, 0) &= \sum_{k \in \mathbb{Z}} \hat{a}(k) e^{ikx} \\ u_t(x, 0) &= \sum_{k \in \mathbb{Z}} \hat{b}(k) e^{ikx} \end{aligned}$$

where $\hat{a}, \hat{b} \in \mathbb{C}$ are given. Want to find \hat{u} . After some work we get

$$\hat{u}(k, t) = \xi(k) \cos(ckt) + \eta(k) \sin(ckt) \quad (4.iv)$$

Adding initial conditions to (4.iv) we obtain

$$\hat{u}(k, t) = \hat{a}(k) \cos(ckt) + \frac{\hat{b}(k)}{ck} \sin(ckt).$$

If $\hat{b}(0) = 0$ then

$$u(x, t) = \sum_{k \in \mathbb{Z}} \left(\hat{a}(k) \cos(ckt) + \frac{\hat{b}(k)}{ck} \sin(ckt) \right) e^{ikx} \quad (4.v)$$

Applying the same idea to the diffusion equation (3.i) $u_t = \kappa u_{xx}$, using initial conditions

$$u(x, 0) = \sum_{k \in \mathbb{Z}} \hat{a}(k) e^{ikx}$$

we obtain that $\dot{\hat{u}}(k, t) = -\kappa k^2 \hat{u}(k, t)$ so

$$u(x, t) = \sum_{k \in \mathbb{Z}} \hat{a}(k) e^{-\kappa k^2 t + ikx}$$

For the inhomogeneous problem $u_k = \kappa u_{xx} + f(x, t)$ with 2π -periodic boundary conditions

$$\hat{u}_t(k, t) = -\kappa k^2 \hat{u}(k, t) + \hat{f}(k, t)$$

we solve to get

$$u(x, t) = \sum_{k \in \mathbb{Z}} \left(e^{-\kappa k^2 t + ikx} \hat{u}(k, 0) + \int_0^t e^{-\kappa k^2 (t-s) + ikx} \hat{f}(k, s) ds \right)$$

4.5.1 Regularity of solutions of homogeneous diffusion equation

Assume that $u(x, 0) \in C^0(\mathbb{R})$. Then $\hat{u}(k, 0) \leq C := \sup_{x \in \mathbb{R}} |u(x, 0)|$.

$$u(x, t) = \sum_{k \in \mathbb{Z}} e^{-\kappa k^2 t + ikx} \hat{u}(k, 0)$$

Proposition 11. u (as defined above) is in $C^\infty(\mathbb{R} \times (0, \infty))$.

Proof. Let $s > 0$. Then

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \underbrace{|\hat{u}(k, 0)|}_{\leq C} |k|^s e^{-\kappa k^2 t} &\leq C \sum_{k=1}^{\infty} |k|^s e^{-\kappa k^2 t} \\ &\leq C \int_1^{\infty} \exp(s \underbrace{\log k}_{< k-1} - \kappa(k-1)^2 t) dk \\ &\leq C \int_1^{\infty} \exp(s(k-1) - \kappa(k-1)^2 t) dk \\ &= C \int_0^{\infty} \exp(sk - \kappa k^2 t) dk \\ &\vdots \\ &\leq \frac{C\sqrt{\pi}}{\sqrt{\kappa t}} \exp\left(\frac{s^2 t}{4\kappa}\right). \end{aligned}$$

By theorem 9, $\sum_{k \in \mathbb{Z}} |\hat{f}(k)| |k|^s < \infty \Rightarrow \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}$ converges to r times differentiable f where $r < s - 1$. Then $u(x, t) \in C^\infty(\mathbb{R} \times (0, \infty))$ as s arbitrary. \square

Proposition 12. u satisfies the diffusion equation.

Proof.

$$\begin{aligned} \frac{\partial}{\partial t} u(x, t) &= \frac{\partial}{\partial t} \sum_{k \in \mathbb{Z}} e^{-\kappa k^2 t + ikx} \hat{u}(k, 0) \\ &= \sum_{k \in \mathbb{Z}} -\kappa k^2 e^{-\kappa k^2 t + ikx} \hat{u}(k, 0) \\ \kappa \frac{\partial^2}{\partial x^2} u(x, t) &= \kappa \frac{\partial^2}{\partial x^2} \sum_{k \in \mathbb{Z}} e^{-\kappa k^2 t + ikx} \hat{u}(k, 0) \\ &= \kappa \sum_{k \in \mathbb{Z}} k^2 e^{-\kappa k^2 t + ikx} \hat{u}(k, 0) \end{aligned} \quad \square$$

4.5.2 Regularity of solutions of homogeneous wave equation

Regularity of solution u (4.v),

$$u(x, t) = \sum_{k \in \mathbb{Z}} \left(\hat{a}(k) \cos(ckt) + \frac{\hat{b}(k)}{ck} \sin(ckt) \right) e^{ikx}$$

does not improve as t increases since $u(x, 0) = u(x, \frac{2\pi}{c})$. Unlike diffusion equation we can solve wave equation backwards in time.

Proposition 13. $u(x, t)$ solves $u_{tt} = c^2 u_{xx}$ if $|\hat{a}(k)| + |\hat{b}(k)| < \frac{c}{|k|^4}$.

Proof. The given condition ensures that u is sufficiently differentiable. It is then a simple check that $\frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u$. \square

4.6 Other boundary conditions

Proposition 14. Let $f \in C^0[0, \pi]$. Can find a_k s.t. $f(x) = \sum_{k=1}^{\infty} a_k \sin(kx)$.

Proof. Extend f to odd 2π -periodic function g . The Fourier coefficients of g given by

$$\begin{aligned} \hat{g}(0) &= \frac{1}{2\pi} \int_0^{2\pi} g(x) dx \\ &\stackrel{g \text{ odd}}{=} 0 \\ \hat{g}(k) &= \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-ikx} dx \\ &\stackrel{\text{symmetry}}{=} \frac{1}{2\pi} \int_0^{2\pi} g(-x) e^{-ikx} dx \\ &\stackrel{\text{periodicity}}{=} -\frac{1}{2\pi} \int_0^{2\pi} g(2\pi - x) e^{ik(2\pi - x)} dx \\ &\stackrel{\text{change vars}}{=} -\frac{1}{2\pi} \int_0^{2\pi} g(y) e^{iky} dy \\ &= -\hat{g}(-k). \end{aligned}$$

This gives us

$$\begin{aligned} \sum_{k \in \mathbb{Z}} e^{ikx} \hat{g}(k) &= \sum_{k=1}^{\infty} (\hat{g}(\cos(kx) + i \sin(kx)) - \hat{g}(k) (\cos(kx) + i \sin(kx))) \\ &= 2 \sum_{k=1}^{\infty} i \hat{g}(k) \sin(kx). \end{aligned}$$

f real valued $\Rightarrow g$ real valued $\Rightarrow \hat{g}(-k) = \overline{\hat{g}(k)}$. Together with $\hat{g}(-k) = -\hat{g}(k)$ we get $\hat{g}(k) \in i\mathbb{R}$.

Then $f(x) = \sum_{k=1}^{\infty} a_k \sin(kx)$ for $x \in (0, \pi)$ where $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx$. \square

Remark 4. If $f(0) = f(\pi)$ then $g \in C^0(\mathbb{R})$, i.e. $\exists a_k$ s.t. $f(x) = \sum_{k=1}^{\infty} a_k \sin(kx)$ for all $x \in \mathbb{R}$.

5 Laplace operator

5.1 Laplace equations

Definition 4 (Laplacian). For $u: \mathbb{R}^n \rightarrow \mathbb{R}$, the *Laplacian* of u , Δu defined as

$$\Delta u(\mathbf{x}) = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(\mathbf{x})$$

5.1.1 Poisson equation

For $\Omega \subset \mathbb{R}^n$ open, boundary $\partial\Omega$ we have the *Poisson equation*

$$\Delta u(\mathbf{x}) = f(\mathbf{x}) \tag{5.i}$$

where $f \in C^0(\Omega)$, with Dirichlet boundary conditions $u(\mathbf{x}) = g(\mathbf{x}) \forall \mathbf{x} \in \partial\Omega$, $g \in C^0(\partial\Omega)$.

5.1.2 Laplace equation

The special case of (5.i),

$$\Delta u(\mathbf{x}) = 0 \tag{5.ii}$$

is called the *Laplace equation*.

Definition 5. Solutions of (5.ii) are called *harmonic functions*.

Example 1. Analytic functions of complex variables are differentiable iff they satisfy Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$. This gives that the solutions are necessarily harmonic, i.e. $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$.

Theorem 15 (Maximum principle). *Let $\Omega \subset \mathbb{R}^n$ open, bounded by $\partial\Omega$. If $u \in C^2(\bar{\Omega})$ and u is harmonic in Ω then u achieves its maximum at some $x \in \partial\Omega$. Alternatively:*

$$\max_{\mathbf{x} \in \bar{\Omega}} u(\mathbf{x}) = \max_{\tilde{\mathbf{x}} \in \partial\Omega} u(\tilde{\mathbf{x}})$$

Proof. Fix $\varepsilon > 0$, let $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon\|\mathbf{x}\|^2$. Then

$$\begin{aligned} \Delta v(\mathbf{x}) &= \Delta(u + \varepsilon\|\mathbf{x}\|^2) \\ &= \underbrace{\Delta u}_{=0} + \varepsilon \underbrace{\Delta\|\mathbf{x}\|^2}_{=2n} \\ &= 2\varepsilon n \\ &> 0 \end{aligned}$$

$\Delta v(\mathbf{x}) \leq 0$ at any interior maximum, so v has no max in Ω° . v continuous, $\bar{\Omega}$ sequentially compact, so by **Differentiation** v achieves a maximum somewhere on $\bar{\Omega}$. Not in interior so max at \mathbf{x}_0 some $\mathbf{x}_0 \in \partial\Omega$.

$\forall \mathbf{x} \in \Omega$,

$$u(\mathbf{x}) < v(\mathbf{x}) \leq v(\mathbf{x}_0) = u(\mathbf{x}_0) + \varepsilon \|\mathbf{x}_0\|^2 \leq \max_{\tilde{\mathbf{x}} \in \partial\Omega} u(\tilde{\mathbf{x}}) + \varepsilon R^2$$

where $R > 0$ s.t. $\Omega \subset \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq R\}$. Ω bounded so $R < \infty$. ε arbitrary so

$$u(\mathbf{x}) \leq \max_{\tilde{\mathbf{x}} \in \partial\Omega} u(\tilde{\mathbf{x}}) \quad \forall \mathbf{x} \in \bar{\Omega} \quad \square$$

5.2 Invariance of Laplace operator

Definition 6. The *special orthogonal group* of order n , $\text{SO}(n) \leq M_n(\mathbb{R})$ s.t. $\forall R \in \text{SO}(n)$

- R orthogonal: $R^T = R^{-1}$
- $\det(R) = 1$.

Remark 5. $\text{SO}(n)$ is the group of rotation matrices.

Proposition 16. *Laplace operator is invariant under the special orthogonal group, i.e. $\Delta_{\mathbf{x}'}u = \Delta_{\mathbf{x}}u$ where $\mathbf{x}' = R\mathbf{x}$, for some $R \in \text{SO}(n)$.*

Proposition 17 (Laplacian in polar co-ordinates). *Let $x = r \cos \theta$, $y = r \sin \theta$. Then*

$$\Delta_2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Are there harmonic functions of the form $u(r, \theta) = f(r)$? They must satisfy

$$0 = r f_{rr} + f_r = (r f_r)_r$$

so $r f_r = c_1$ and $f(r) = c_1 \log r + c_2$.

Proposition 18 (Laplacian in spherical co-ordinates). *Let $x = r \cos \varphi \sin \theta$, $y = r \sin \varphi \sin \theta$, $z = r \cos \theta$. Then*

$$\Delta_3 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2} \left(u_{\theta\theta} + (\cot \theta)u_\theta + \frac{1}{\sin^2 \theta} u_{\varphi\varphi} \right)$$

Are there rotationally invariant harmonic functions of the form $u(r, \theta, \varphi) = f(r)$? They must satisfy

$$f_{rr} + \frac{2}{r}f_r = 0 \iff (r^2 f_r)_r = 0$$

so $f_r = \frac{c_1}{r^2}$ and $f(r) = -\frac{c_1}{r} + c_2$.

5.3 Solutions of Laplace equation (5.ii) via Fourier series

Example 2. $D = (0, \pi) \times (0, \pi) \times (0, \pi)$, boundary conditions

$$u(0, y, z) = u(x, 0, z) = u(x, y, 0) = u(x, \pi, z) = u(x, y, \pi) = 0$$

$$u(\pi, y, z) = g(y, z).$$

We seek solutions of the form $u(x, y, z) = a(x)b(y)c(z)$.

$$\begin{aligned} \Delta_3 u = 0 &\iff a''bc + ab''c + abc'' = 0 \\ &\iff \frac{a''}{a}(x) + \frac{b''}{b}(y) + \frac{c''}{c}(z) = 0. \end{aligned}$$

Boundary conditions then become

$$a(0) = b(0) = c(0) = b(\pi) = c(\pi) = 0$$

We need each quotient to be constant, i.e.

$$\frac{a''}{a} = \nu, \quad \frac{b''}{b} = \lambda, \quad \frac{c''}{c} = \mu$$

Combining with the boundary conditions we get $b(y) = B \sin(my)$, $c(z) = C \sin(nz)$ where $\lambda = -m^2$, $\mu = -n^2$.

$a'' = \nu a$, $a(0) = 0$ so $\nu a = m^2 + n^2 > 0$ gives $a(x) = A \sinh(\sqrt{m^2 + n^2}x)$.

Summing up we get

$$u(x, y, z) = \sum_{m,n=1}^{\infty} A_{m,n} \sinh(\sqrt{m^2 + n^2}x) \sin(my) \sin(nz)$$

Where $A_{m,n}$ found by expanding g into sine series for each y , we get

$$A_{m,n} = \frac{4}{\pi^2 \sinh(\sqrt{m^2 + n^2}\pi)} \int_0^\pi \int_0^\pi g(y, z) \sin(my) \sin(nz) dy dz$$

5.4 Sub-mean value property of subharmonic functions

Definition 7. $u: \Omega \rightarrow \mathbb{R}$ is *subharmonic* if $\Delta u(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \Omega$.

Example 3. Strictly convex functions, i.e. those s.t.

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y) \quad \forall x, y \in \mathbb{R}, x \neq y, t \in [0, 1]$$

are subharmonic.

Definition 8. The (*surface*) *mean value* $\bar{u}_R(\mathbf{x}_0)$ of u on $B(\mathbf{x}_0, R) \subset \Omega$ defined by

$$\bar{u}_R(\mathbf{x}_0) = \frac{1}{4\pi R^2} \int_{\partial B(\mathbf{x}_0, R)} u dS$$

Definition 9. $u: \Omega \rightarrow \mathbb{R}$ has *sub-mean value* property if whenever $\overline{B(\mathbf{x}_0, R)} \subset \Omega$ then $u(\mathbf{x}_0) \leq \bar{u}_R(\mathbf{x}_0)$.

Remark 6. This says value of u at centre of ball \leq average on boundary.

Theorem 19. Let $\Omega \subset \mathbb{R}^n$ open. If $u \in C^2(\Omega)$, $\Delta u \geq 0$ (i.e. u is subharmonic) and $\overline{B(\mathbf{x}_0, R)} \subset \Omega$, then u has sub-mean value property, i.e. $u(\mathbf{x}_0) \leq \bar{u}_R(\mathbf{x}_0)$.

Proof. IDEA: USE GREEN'S THEOREM AND POLAR CO-ORDINATES.

Green's identity: $w: E \rightarrow \mathbb{R}$. Then

$$\int_E \nabla w \cdot \nabla f dV = \int_{\partial E} f \frac{\partial w}{\partial n} dS - \int_E f \Delta w dV \quad (5.iii)$$

Use $f \equiv 1$, $E = B(\mathbf{x}_0, R)$ in (5.iii). Then $\nabla f = 0$ so

$$\int_{\partial B(\mathbf{x}_0, R)} \frac{\partial w}{\partial n} dS = \int_{B(\mathbf{x}_0, R)} \Delta w dV \geq 0 \quad (5.iv)$$

Polar co-ordinates centred at \mathbf{x}_0 :

$$(x, y, z) = (x_0 + r \cos \varphi \sin \theta, y_0 + r \sin \varphi \sin \theta, z_0 + r \cos \theta)$$

$$\begin{aligned} \left. \frac{d}{dr} u(x_0 + r \cos \varphi \sin \theta, y_0 + r \sin \varphi \sin \theta, z_0 + r \cos \theta) \right|_{r=R} \\ &= \frac{\partial u}{\partial x}(\mathbf{x}) \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}(\mathbf{x}) \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}(\mathbf{x}) \frac{\partial z}{\partial r} \\ &\stackrel{\text{Chain rule}}{=} \nabla u(x_0 + r \cos \varphi \sin \theta, y_0 + r \sin \varphi \sin \theta, z_0 + r \cos \theta) \cdot \mathbf{n} \\ &= \frac{\partial u}{\partial n}. \end{aligned}$$

For $r > 0$,

$$\begin{aligned} \frac{d}{dr} \bar{u}_r(\mathbf{x}_0) &= \frac{d}{dr} \left(\frac{1}{4\pi r^2} \int_{\partial B(\mathbf{x}_0, r)} u dS \right) \\ &= \frac{1}{4\pi r^2} \int_0^{2\pi} \int_0^\pi \frac{\partial u}{\partial n}(x_0 + r \cos \varphi \sin \theta, y_0 + r \sin \varphi \sin \theta, z_0 + r \cos \theta) dS \end{aligned}$$

where $dS = r^2 \sin \theta d\theta d\varphi$.

This gives $\frac{d}{dr} \bar{u}_r(\mathbf{x}_0) \geq 0$ by (5.iv). Therefore if $0 < \sigma < R$ then

$$\bar{u}_\sigma(\mathbf{x}_0) \leq \bar{u}_R(\mathbf{x}_0) \quad (5.v)$$

Show that $\lim_{\sigma \downarrow 0} \bar{u}_\sigma(\mathbf{x}_0) = u(\mathbf{x}_0)$.

Given $\varepsilon > 0 \exists \delta > 0$ s.t.

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta \Rightarrow |u(\mathbf{x}) - u(\mathbf{x}_0)| < \varepsilon$$

so $0 < \sigma < \delta \Rightarrow$

$$\begin{aligned}
|u(\mathbf{x}_0) - \bar{u}_\sigma(\mathbf{x}_0)| &= \left| \frac{1}{4\pi\sigma^2} \int_{\partial B(\mathbf{x}_0, \sigma)} u(\mathbf{x}_0) - u(\mathbf{x}) dS \right| \\
&\leq \frac{1}{4\pi\sigma^2} \int_{\partial B(\mathbf{x}_0, \sigma)} |u(\mathbf{x}_0) - u(\mathbf{x})| dS \\
&< \frac{1}{4\pi\sigma^2} \int_{\partial B(\mathbf{x}_0, \sigma)} \varepsilon dS \\
&= \varepsilon. \quad \square
\end{aligned}$$

Remark 7. If u harmonic then u has *mean value property*, $u(\mathbf{x}_0) = \bar{u}_R(\mathbf{x}_0)$ whenever $\overline{B(\mathbf{x}_0, R)} \subset \Omega$. This is because both u and $-u$ are subharmonic.

Theorem 20 (Strong maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded. Let $u \in C^2(\Omega)$ be harmonic. Then u achieves its maximum in Ω iff u is constant.*

Proof. Let $\mathbf{x}_m \in \bar{\Omega}$ point where u maximised. Assume $\mathbf{x}_m \in \Omega$. Find $\varepsilon > 0$ s.t. $B(\mathbf{x}_m, \varepsilon) \subset \Omega$. Then average $\leq \max$ so

$$M := u(\mathbf{x}_m) \underset{\text{mean value}}{=} \bar{u}_\varepsilon(\mathbf{x}_0) \leq M$$

so $u(\mathbf{x}) = M \forall \mathbf{x} \in B(\mathbf{x}_m, \varepsilon)$.

Repeat with new centre. Can fill Ω with spheres. As Ω connected, $u(\mathbf{x}) = M$ throughout. \square

Theorem 21 (Dirichlet's principle). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $u \in C^1(\bar{\Omega})$ be harmonic and subject to the boundary condition $u(\mathbf{x}) = f(\mathbf{x}) \forall \mathbf{x} \in \partial\Omega$. Then for $v \in C^1(\bar{\Omega})$ satisfying same boundary condition then*

$$\int_{\Omega} \|\nabla u(\mathbf{x})\|^2 d\mathbf{x} \leq \int_{\Omega} \|\nabla v(\mathbf{x})\|^2 d\mathbf{x}.$$

Proof. Let $w = u - v$. Then $w \in C^1(\bar{\Omega})$ and $w(\mathbf{x}) = 0 \forall \mathbf{x} \in \partial\Omega$.

$$\begin{aligned}
\int_{\Omega} \|\nabla v\|^2 d\mathbf{x} &= \int_{\Omega} \|\nabla v + \nabla w - \nabla w\|^2 d\mathbf{x} \\
&= \int_{\Omega} \|\nabla u\|^2 d\mathbf{x} - 2 \int_{\Omega} \langle \nabla u, \nabla w \rangle d\mathbf{x} + \int_{\Omega} \|\nabla w\|^2 d\mathbf{x} \\
&\stackrel{\text{parts}}{=} \int_{\Omega} \|\nabla u\|^2 d\mathbf{x} + 2 \int_{\Omega} \underbrace{w(\mathbf{x}) \operatorname{div}(\nabla u(\mathbf{x}))}_{=\Delta u=0} d\mathbf{x} \\
&\quad - \int_{\partial\Omega} \underbrace{w(\mathbf{x})}_{=0 \text{ on } \partial\Omega} \nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) dS + \int_{\Omega} \underbrace{\|\nabla w\|^2}_{\geq 0} d\mathbf{x} \\
&\geq \int_{\Omega} \|\nabla u\|^2 d\mathbf{x}. \quad \square
\end{aligned}$$

5.5 Types of second order equations

Theorem 22. *By linear transformation of variables x and y can reduce the equation*

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0 \quad (5.vi)$$

into one of the following three forms:

1. *If $a_{12}^2 < a_{11}a_{22}$ it can be transformed into an elliptic equation*

$$u_{xx} + u_{yy} + \text{l.o.t.} = 0$$

2. *If $a_{12}^2 > a_{11}a_{22}$ it can be transformed into an hyperbolic equation*

$$u_{xx} - u_{yy} + \text{l.o.t.} = 0$$

3. *If $a_{12}^2 = a_{11}a_{22}$ it can be transformed into an parabolic equation*

$$u_{xx} + \text{l.o.t.} = 0$$