MA250 Introduction to Partial Differential Equations

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Contents

Lecture 1: 8/1/07

0 Notation and Fundamental concepts

If u is a function of more than one variable then $u = u(x, y, \ldots)$

0.1

We will often denote the the partial derivatives of u by subscripts so for example:

$$\frac{\partial u}{\partial x} = u_x \quad \frac{\partial u}{\partial y} = u_y$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx}$$

Partial differential equations (PDE's) are equations which involve $u, x, y, \ldots$ and the partial derivatives of these, $u_x, u_y, u_{xy}, \ldots$.

$$F(x, y, \ldots, u, u_x, u_y, \ldots) = 0 \tag{0.1}$$

If F depends only on $x, y, \ldots, u, u_x, u_{xx}, u_{xxx}, \ldots$ i.e the partial derivatives with respect to $y, z, \ldots$ do not appear in F then equation 0.1 is an Ordinary Differential Equation (ODE). If the function F depends on $x, y, \ldots, u, u_x, u_y, \ldots$ but not on higher derivatives $u_{xx}, u_{xy}, u_{yy}, \ldots$ then F is a first-order PDE. If F only depends on $x, y, \ldots, u$ the 1st and 2nd order partial derivatives $(u_x, u_y, \ldots, u_{xx}, u_{xy}, u_{yy})$ but not higher derivatives, then in is a second-order PDE.

Example 0.1 (Some examples of PDE's)**.**

$$u_x + u_y = 0 \quad \text{Transport equation} \tag{0.2}$$

$$u_x + yu_y = 0^1 \tag{0.3}$$

$$u_x + uu_y = 0 \quad \text{Burgers}' \text{ equation} \tag{0.4}$$

$$u_{xx} + u_{yy} = 0 \quad \text{Laplace equation} \tag{0.5}$$

$$u_{tt} = u_{xx} + u \quad \text{Wave equation} \tag{0.6}$$

$$u_t + u_{xxx} - 6uu_x = 0 \quad \text{Korteweg de Vries equation} \tag{0.7}$$

$$u_{tt} + u_{xxxx} = 0 \quad \text{Vibrating Bar equation} \tag{0.8}$$

$$u_t - u_{xx} = 0, \quad \text{Schrödinger equation} \tag{0.9}$$

We say that a PDE is linear if and only if F depends linearly on $u, u_x, u_y, u_{xx}, \ldots$ Equations 0.2, 0.3, 0.5, 0.8, 0.9 in example 0.1 are linear, the other equations are non-linear.

[1]Sadly as far as I know this equation has no name, it must feel very sad, if you can find a name for it give me an email.

In this course we will only study first and second order linear ODE's. We will show for certain equations that solutions exist and constant explicit formulas in simple cases. In may cases you cannot solve a PDE analytically and must instead solve the PDE numerically. Additionally we will also show that the solutions are unique and the properties of the solutions depend in a significant way on the PDE. We will at the end of the course classify some PDE's and demonstrate that the qualitative properties of the solutions are similar for equations with the same type.

0.2 Important results from Analysis

0.2.1 Derivatives are local

Derivatives are local, i.e. $\frac{\delta u}{\delta x}(x_0, y_0)$ is completely determined by the values of u near (x_0, y_0).

0.2.2 Derivatives commute

Derivatives commute if u is smooth[2] i.e. $u_{xy} = u_{yx}$.

0.2.3 Chain Rule

For some arbitrary C^1 functions f and g, the chain rule states that:

$$\frac{\partial}{\partial x}\big[f(g(x,t))\big] = f'(g(x,t))\frac{\partial g}{\partial x}(x,t) \tag{0.10}$$

0.2.4 Derivatives of integrals

We need to be able to find derivatives of integrals like

$$I(t) = \int_{a(t)}^{b(t)} f(x,t)dt$$

0.2.5 Divergence theorem

We need to be able to use the divergence theorem which we covered in Vector Analysis.

$$\int_{\Omega} \nabla \dot{V} dV = \int_{\delta\Omega} V \cdot \hat{N} dS \tag{0.11}$$

[2]This is a very interesting statement, some people say that smooth means C^∞, but generally it is used to mean that the function can be differentiated enough times for the problem to be solved, or in this case for the result to be true, which requires the function to be C^1. This is proved in part 4 of MA225 (differentiation).

0.2.6 Transforming coordinates

For a function f, in x, y coordinates, we can transform it into x', y' co-ordinates by using

$$\int_{\Omega} f(x,y)dxdy = \int_{\Omega'} f\big(g(x',y')\big)h(x',y')J(x',y')dx'dy'$$

where J is the Jacobian determinant which in this case is:

$$J(x,y) = \left| \frac{\partial g}{\partial x'}\frac{\partial h}{\partial y'}(x',y') - \frac{\partial g}{\partial g'}\frac{\partial h}{\partial x'}(x',y') \right|$$

0.2.7 Directional derivatives

We also need to know the directional derivative which for a function u in direction v is defined as:

$$D_{\boldsymbol{v}}u(a) = \frac{\partial u}{\partial v}(a) = \lim_{h\to 0}\frac{u(a+hv)-u(a)}{h} = \nabla u(a)\cdot v \tag{0.12}$$

Lecture 2: 9/1/07

Example 0.2. Find all the functions $u(x,y)$ which satisfy the PDE $u_{xx} = 0$[3] The FTC implies that $u_x(x,y) = a(y)$. Applying this argument once more gives a general solution of the form of $u(x,y) = xa(y) + b(y)$, where a and b are arbitrary functions.

Example 0.3. $u_{xy} = 0$ Integrate this with respect to x gives us $u_y = a(y)$, then integrating this result with respect to y gives us $u = A(y) + B(x)$ where $A' = a$. This means that A is the anti-derivative of a.

Here we can see that the solutions of PDE's depend on arbitrary functions not just constants as we have done when solving differential equations before. We will see that these arbitrary functions are determined by the initial conditions and the boundary conditions of the problem we are trying to solve with the PDE.[4]

1 First Order Linear equations

1.1 Derivation

Let $u(x,t) \in [0,\infty]$ be the concentration of a substance of interest

Example 1.1. Number of cars per kilometre of road. Assume that the cars move with a constant velocity $c \in \mathbb{R}$, this is shown in figure 1 and implies that:

$$u(t+h,x) - u(t,x) = -(u(x,t) - u(x-ch,t)$$

Divide by h.

[3]This is a one dimensional version of the Laplace Equation, which is covered in more detail in section 5.

[4]For example by using different boundary conditions allow Laplace's equation to be used for solving a wide range of problems including heat distribution, fluid flow and electric conduction.

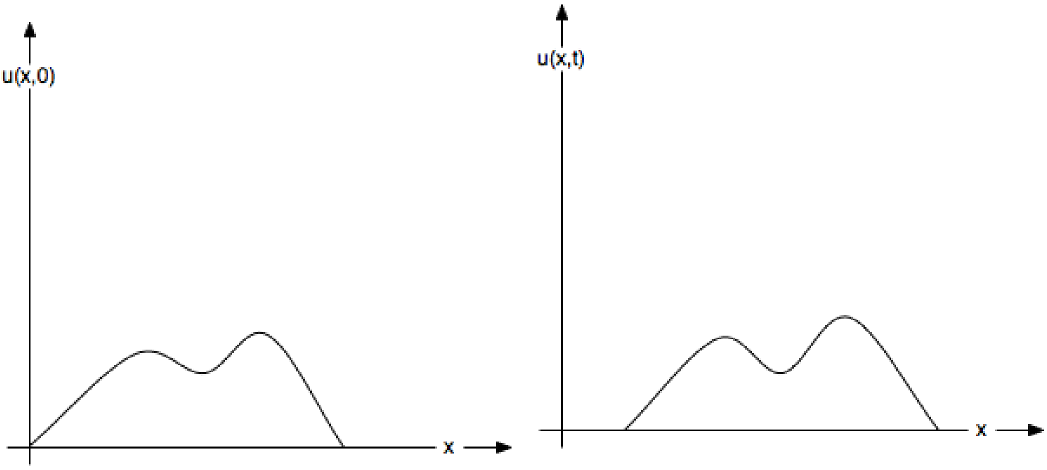

Figure 1: Graphs showing a function u at a time of $t' = 0$ and $t' = t$.

$$\frac{1}{h}[u(t+h,x) - u(t,x)] = \frac{-1}{h}[u(x,t) - u(x-ch,t)]$$

Now we take the limit as $h \to 0$ on both sides and obtain that $u_t = -cu_x$. This is the simple one dimensional transport equation.[5]

Remark 1.1. *The transport velocity does not depend on x, t or u. In the heterogeneous case where $c = c(x,t)$ and c is no longer a constant, the previous argument doesn't work any more and we have to think a bit harder. We are interested in the evaluation of the integral*

$$M(t) = \int_a^b u(x,t)dx \tag{1.1}$$

Furthermore we can find the change of $M(t)$ by measuring the flux through the boundary of the interval $[a,b]$.

$$J(t) = c(a,t)u(a,t) - c(b,t)u(b,t)$$

Then by taking the limits as $a = x$, $b = x+h$, we then get

$$\frac{d}{dt}\int_x^{x+h} u(s,t)ds = c(x,t)u(x,t) - c(x+h,t)u(x+h,t)$$

$$\frac{1}{h}\int_x^{x+h} u_t(s,t)ds = \frac{-1}{h}\left[c(x+h,t)u(x+h,t) - c(x,t)u(x,t)\right]$$

Finally if $u(x,t)$ is differentiable then by taking on both sides the limit as $h \to 0$ we obtain:

$$u_t = -(cu)_x \Leftrightarrow u_t + cu_x + c_x u = 0$$

[5]see equation 0.2 in example 0.1

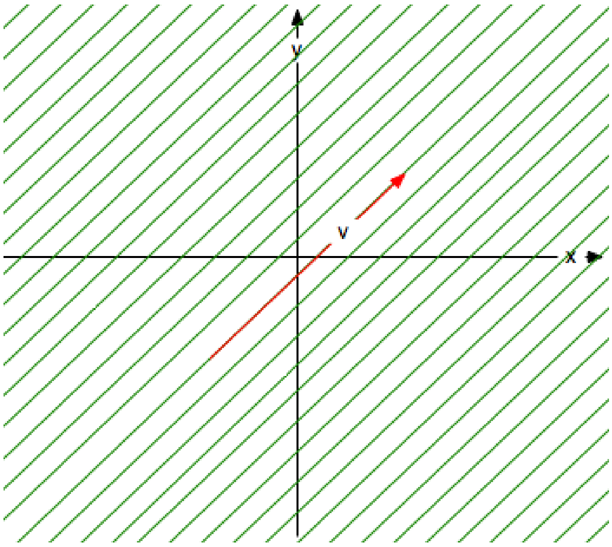

Figure 2: Graph showing a series of diagonal lines and direction vector v

1.2 Method of characteristics

Let us solve the following PDE where $a, b \neq 0$ and where a and b are constant.

$$au_x + bu_y = 0 \tag{1.2}$$

For example if $a = 1$ and $b = 0$ then $u(x, y) = f(y)$ where f is any function of one variable.

1.2.1 Geometric method

We can solve equation 1.2 using the coordinate method. If $au_x + bu_y$ is the directional derivative of the function u at (x, t) in the direction $v = (a, b)$. This implies that u is constant in the direction v. An example of this is shown in figure 2.

Lecture 3: 11/1/07 Assume that without loss of generality that $b \neq 0$, Every point (x, y) can then be written in the form of $(x, y) = (x_0 + ta, tb)$, where $t = \frac{y}{b}$ and $x_0 = x - y\frac{a}{b}$. This implies that

$$u(x, y) = f(x_0) = f\left(x - y\frac{a}{b}\right) \tag{1.3}$$

is the complete solution of the equation 1.2 where f is a function of one variable.

1.2.2 Second Method: Coordinate Method

We can also solve equation 1.2 by the coordinate method, first we do a change of variables to $x' = ax + by$, $y' = bx - ay$. We also have to replace the derivatives with respect to x and y by derivatives with respect to x' and y'. By the chain rule:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \tag{1.4}$$

Then with respect to y we also find that:

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \tag{1.5}$$

Hence we obtain that $au_x + bu_y = 0$ if and only if

$$a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = 0 \Leftrightarrow (a^2 + b^2)u_{x'} = 0$$

and then equation 1.2[6] becomes:

$$u_{x'} = 0$$

with respect to the primed variables. Hence

$$u(x', y') = u(0, bx - ay) = g(bx - ay) = f\left(x - y\frac{a}{b}\right) \tag{1.6}$$

Where $f(t) = g(bt)$. As you can see the results obtained from equations 1.3 and 1.6 are identical.

1.2.3 The variable co-efficient case

We can obtain better understanding of this method by considering the more general case where a and b aren't constant and can depend on x and y but not on u. Note that in this case equation 1.2 doesn't describe the heterogeneous transport equation, the correct equation would therefore be

$$(au)_x + (bu)_y = 0^7 \tag{1.7}$$

Example 1.2. If $a = 1$ and $b = y$ we would to solve the PDE $u_x + yu_y = 0$ (equation 0.3). Using the geometric method again we can deduce that the solutions of u are constant in the direction $v(x, y) = (1, y)$. As can be seen in figure 3, the curves $\big(x, y(x)\big)$ with tangent vectors $(1, y)$ fill the x, y plane without intersection. The solutions of equation 0.3 imply that

$$\frac{dy}{dx} = \frac{y}{1} = \frac{b}{a} \tag{1.8}$$

Clearly for some $C \in \mathbb{R}$ $y(x) = Ce^x$ is valid solution for equation 1.8. These curves are called the characteristic curves of the PDE 0.3. As the value of the constant C is changed the curves (x, Ce^x) fill the x, y plane. On each curve u is constant since we can see that:

$$\frac{\partial}{\partial x}u(x, Ce^x) = \frac{\partial u}{\partial x} + Ce^x\frac{\partial u}{\partial y} = 0$$

which clearly holds for all functions which satisfy 0.3 ($u_x + yu_y = 0$).

$$u(x, Ce^x) = u(0, Ce^0) = u(0, C)$$

[6] In case you have forgotten this is $au_x + bu_y = 0$

[7] Someone is borrowing (the horror!) the Strauss book at my favourite library (*cough* Radcliffe Science Library, `http://www.ouls.ox.ac.uk/rsl` *cough*), and my copy is at home so I can't check that it is $(au)_x$, (it's not in my original notes) though that seems highly likely and makes sense that way.

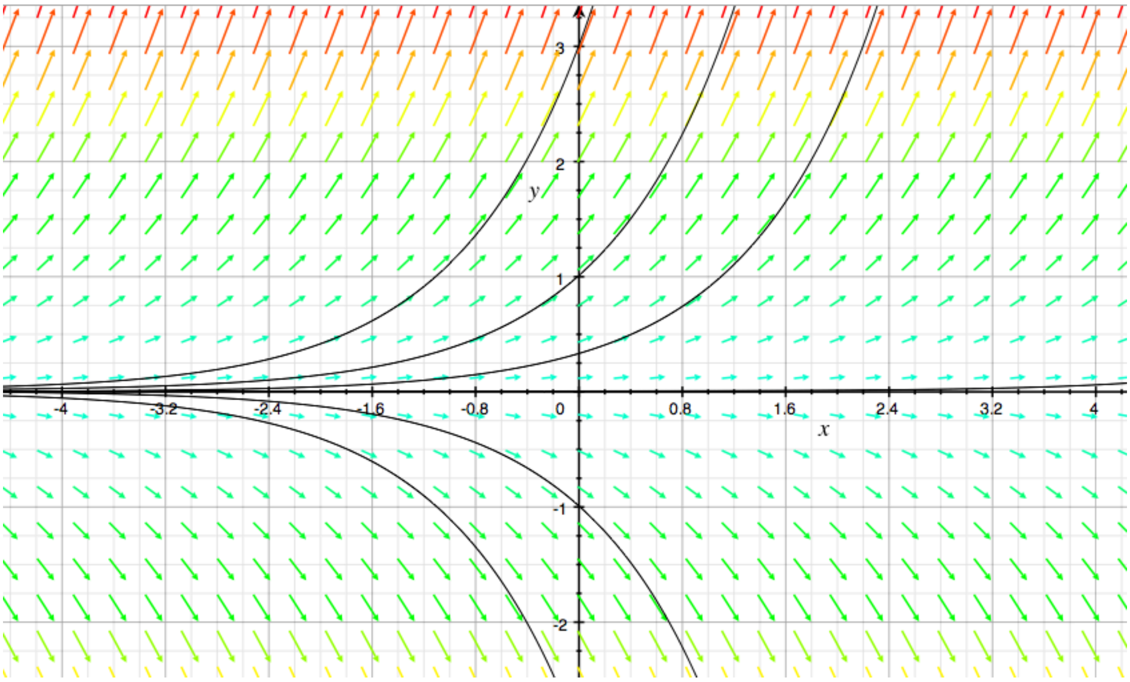

Figure 3: Graph of the vector field $(1, y) = v(x, y)$, with some examples of tangent vectors.

This means that u is independent of x. Taking $y = Ce^x$ we therefore find that $C = e^{-x}y$ we have found that where f is a function of 1 variable.

$$u(x, y) = f(e^{-x}y) \tag{1.9}$$

Is the general solution of equation 0.3.

Remark 1.2. *There is no hope to find a general solution for all linear first order equations with variable co-efficients since there is no solution formula for all first order ODE's.*

Lecture 4: 15/1/07

Remark 1.3. *We can solve the transport equation* $au_x + bu_y = 0$ *by integrating the ODE*

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$$

Solutions of PDE's are not necessarily smooth. In our examples u *is as smooth as* f *so for example if* f *is discontinuous, then* u *is also discontinuous.*

2 The Wave equation

Where the constant C is the wave speed

$$u_{tt} = C^2 u_{xx} \tag{2.1}$$

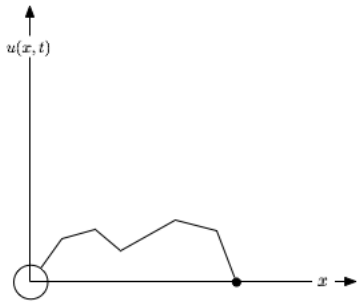

Figure 4: Graph of $u(x, t)$

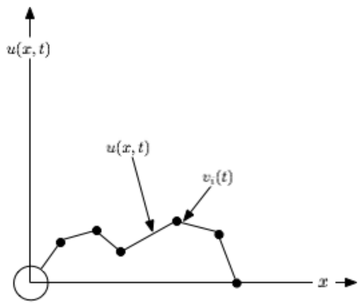

Figure 5: Second graph of $u(x, t)$ with labels

$u(x, t)$ can be interpreted as the vertical displacement of a vibrating spring. Equivalently where $\rho > 0$ is the mass density of the string and $T > 0$ is the elastic constant this equation can be written as:

$$\rho u_{tt} = T u_{xx} \tag{2.2}$$

this equation shows how much force the string exerts if stretched. Equations 2.1 and 2.2 are equivalent if $C = \sqrt{\frac{T}{\rho}}$.

2.1 Derivatives

Example 2.1. Consider an interval $[0, 1]$ and a vibrating string which with ends attached to $(0, 0)$ and $(1, 0)$, this is shown in figure 4.

In order to determine the equations of motion we assume that $u(x, t)$ represents the positions of N particles that interact with each other by linearly elastic springs, a diagram of this is shown in figure 5 [8]

If we say that $q_i(t) \in \mathbb{R}^2$ is the position of the i_{th} particle at time t, then the evolution of the particle satisfies Newton's equation of motion.

$$m\ddot{\boldsymbol{q}}_i = \boldsymbol{f}_i$$

[8] This is very similar to figure 4 except that it is labelled

This is where $\boldsymbol{f}_i \in \mathbb{R}^2$ is the force exerted by the springs on particle i therefore since the springs are linearly elastic therefore for $k > 0$ where k is the spring constant, we have the following:

$$\boldsymbol{f}_i = k(\boldsymbol{q}_{i+1} - \boldsymbol{q}_i) + \tau(\boldsymbol{q}_{i-1} - \boldsymbol{q}_i) \tag{2.3}$$

Now we can deduce the wave equation from this, first we assume that $q_i = \left(\frac{i}{N}, v_i\right)$ where $v_i \in \mathbb{R}$ is the vertical displacement of particle i. Under this assumption we can say that:

$$f_i = (k\frac{1}{n}(i - 1 - 2i + i + 1), \tau(v_{i+1} - 2v_i + v_{i-1})) = (0, k(v_{i+1} - 2v_i + v_{i-1})) \tag{2.4}$$

This leads to the scalar differential equation

$$m\ddot{v}_i = k(v_{i+1} - 2v_i + v_{i-1}) \tag{2.5}$$

Now we can consider the (thermodynamic) limiting behaviour as $N \to \infty$. Assume that

$$v_i(t) = u\left(\frac{i}{N}, t\right) \tag{2.6}$$

where u is a smooth function of x and t.

$$k = \frac{T}{h^2} \tag{2.7}$$

Then by putting equations 2.6 and 2.7 into equation 2.5 we obtain that:

$$\frac{d\ddot{v}}{dt^2} = \frac{T}{h^2}\left(u\left(\frac{i+1}{N}, t\right) - 2u\left(\frac{i}{N}, t\right) + u\left(\frac{i-1}{N}, t\right)\right)$$

where $h = \frac{1}{N}$

$$= \frac{T}{h^2}\left[u\left(\frac{i}{N}\right) + hu_x\left(\frac{i}{N}\right) + \frac{1}{2}h^2 u_{xx}\right]$$

$$= Tu_{xx}\left(\frac{i}{N}, t\right) + O(1)$$

So sending N to ∞ yields that.

$$m\delta_t^2 u(x,t) = Tu_{xx}(x,t) \tag{2.8}$$

where $x = \frac{i}{N}$.

Definition 2.1 (Big O notation 1)**.** *We say that $f(x)$ is $O\big(g(x)\big)$ as $x \to \infty$ if and only if $\exists x_0, M$ s.t for $x > x_0$*

$$|f(x)| \leq M|g(x)|$$

Similarly as $x \to 0$, $\exists \delta, M > 0$ s.t for $x < \delta$

$$|f(x)| \leq M|g(x)|$$

[4]

Definition 2.2 (Big O notation 2)**.** *These can be combined into one definition: $f(x)$ is $O\big(g(x)\big)$ as $x \to a$ if*

$$\limsup_{x\to a} \left|\frac{f(x)}{g(x)}\right| < \infty$$

Note that in this case $a = 0, \infty$.[4]

Lecture 5: 16/1/07

Remark 2.1. *Equation 2.8 then leads to:*

$$mu_{tt} = Tu_{xx} \tag{2.9}$$

(We can replace m by mass density ρ in equation 2.9) The rigorous justification of the derivation is an interesting mathematical challenge which often leads to unexpected insights.

2.2 Construction of solutions of the Wave equation on the real line

First consider the simplest possible setting without boundary. i.e. $u = u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$. This corresponds to a very long string. The reason why this scenario is so simple is the identity

$$u_{tt} - C^2 u_{xx} = \left(\frac{\partial}{\partial t} - C\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + C\frac{\partial}{\partial x}\right) u = 0 \tag{2.10}$$

We introduce the new coordinates of $\xi = x + Ct$ and $\eta = x - Ct$[9] The chain rule implies that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}$$

and similarly

$$\frac{\partial}{\partial t} = C\frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} - C\frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$\frac{\partial}{\partial t} - C\frac{\partial}{\partial x} = -2C\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} + C\frac{\partial}{\partial x} = 2C\frac{\partial}{\partial \xi}$$

which means that

$$u_{\eta\xi} = 0 \tag{2.11}$$

Since $C \neq 0$ the solution of equation 2.11 is given by

$$u(\xi, \eta) = f(\xi) + g(\eta)$$

for two arbitrary functions f and g. In x, t coordinates u then becomes

$$u(x,t) = f(x + Ct) + g(x - Ct) \tag{2.12}$$

[9] Applied mathematicians love ξ and η, they show up a lot in this area of mathematics.

2.2.1 The initial value problem

In most cases we are interested in the question of how the solution $u(x, t)$ of the wave equations depends on the following initial conditions.

$$\begin{aligned} u(x, 0) &= \phi \\ u_t(x, 0) &= \psi \end{aligned} \tag{2.13}$$

The solution of the wave equation, 2.13 is easily found using equation 2.12

$$f + g = \phi \tag{2.14}$$

$$Cf' - Cg' = \psi \tag{2.15}$$

Differentiating equation 2.14 with respect to x yields

$$\phi' = f' + g' \tag{2.16}$$

Then dividing equation 2.15 by C, yields

$$\frac{1}{C}\psi = f' - g' \tag{2.17}$$

Adding and subtracting equations 2.16 and 2.17 then gives

$$f' = \frac{1}{2}\left(\phi' + \frac{1}{C}\psi\right)$$

$$g' = \frac{1}{2}\left(\phi' - \frac{1}{C}\psi\right)$$

Integration of these two equations then leads to

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2C}\int_0^s \psi(r)dr + A$$

$$g(s) = \frac{1}{2}\phi(s) + \int_0^s \psi(r)dr + B$$

where A and B are arbitrary constants. Since we know that $f + g = \phi$ from equation 2.14, we have that $A + B = 0$ and we obtain

$$u(x, t) = \frac{1}{2}\left[\phi(x + Ct) + \phi(x - Ct) + \frac{1}{C}\int_0^{x+Ct} \psi - \frac{1}{C}\int_0^{x-Ct} \psi\right]$$

$$u(x, t) = \frac{1}{2}\left[\phi(x + Ct) + \phi(x - Ct) + \frac{1}{C}\int_{x-Ct}^{x+Ct} \psi(s)ds\right] \tag{2.18}$$

This is the solution formula for the initial value problem found by d'Alembert in 1746. Assuming that ϕ and ψ are in $C^2(\mathbb{R})$ we can deduce from d'Alembert's formula that it gives a bona fide solution of the wave equation.

Example 2.2. For $\phi(x) = 0$ and $\psi(x) = \cos x$ the solution of the wave equation is

$$u(x, t) = \frac{1}{2C}\big(\sin(x + Ct) - \sin(x - Ct)\big) = \frac{1}{C}\cos x \sin(Ct) \tag{2.19}$$

This is known as the standing wave equation.

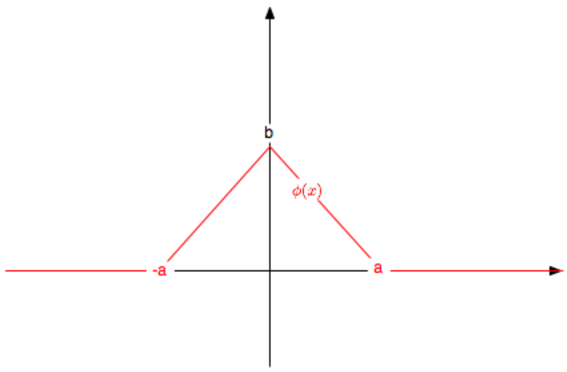

Figure 6: First graph of plucked string

Lecture 6: 18/1/07

Example 2.3. Recall the conditions for the wave equation

$$u_{tt} = cu_{xx},\ u(x,0) = \phi(x),\ \psi(x) = u_t(x,t)$$

Also recall equation 2.18.

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\frac{1}{c}\int_{x-ct}^{x+ct}\psi(s)ds\right]$$

Now consider an infinitely long plucked string with initial position

$$\phi(x) = \begin{cases} b - \frac{b(x)}{a} & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

and initial velocity $\psi(x) = 0\ \forall x$. As you can see in figure 6 this corresponds to a three finger pluck with all three fingers removed at the same time. [10]

Remark 2.2. *Writing the formulae that correspond to the figures 6 and 7 is considerably more work than drawing the pictures.*

2.3 Causality and energy conservation

2.3.1 Causality

D'lembert's solution

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds \tag{2.20}$$

Shows in a rather explicit way that solutions of the wave equation satisfy the *Causality Principle*. In u the value of the initial position (ϕ) and initial velocity (ψ) at x_0 affects the solution $u(x,t)$ only for $x \in [x_0 - ct, x_0 + ct]$. As a consequence of figure 8 if $\psi(x) = \phi(x) = 0$ for $|X| > R$ then $u(x,t) = 0$ for all $|x| > R + ct$ An inverse way to express locality is shown in the following example

[10] Can someone with some musical knowledge please correct this section and email me, thanks!

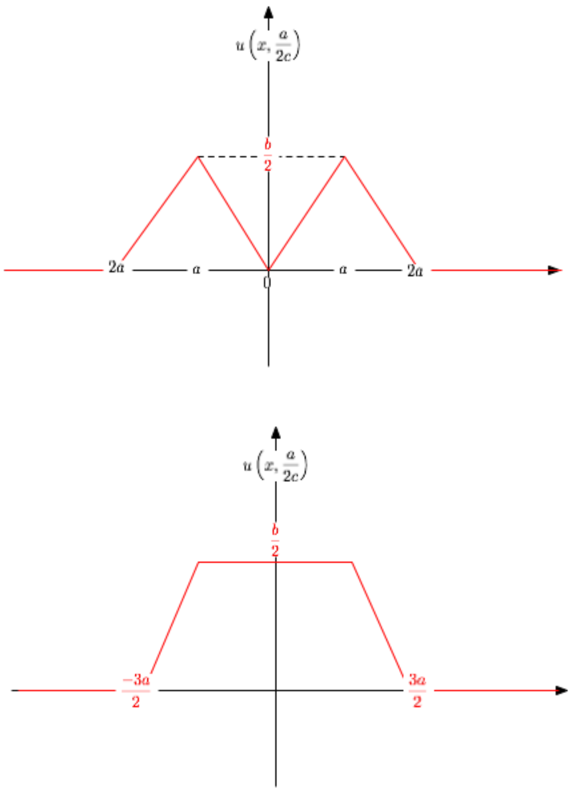

Figure 7: Second and third graph of plucked string

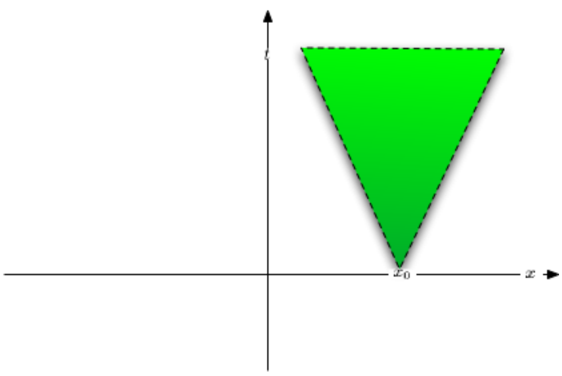

Figure 8: Domain of influence

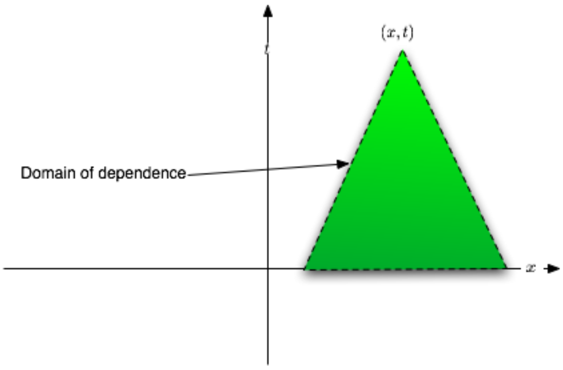

Figure 9: Domain of dependence

Example 2.4. Fix (x, t), the values of $u(x, t)$ then depend on $\phi(y), \psi(y)$ for $|x - y| \leq ct$.

2.3.2 Energy conservation

As in the previous subsection let $u(x, t)$ be a solution of the wave equation ($\rho u_{tt} = T u_{xx}$) which has the property that $u(x, t) = 0 \forall |x| \leq R, |t| \leq S$[11]. This can be achieved by choosing ϕ and ψ such that for some constant c that

$$\phi(x) = \psi(x) = 0 \forall |x| \leq R - cS$$

We then define $E(t) = K(t) + P(t)$ where:

$$K(t) = \frac{\rho}{2} \int_{-R}^{R} u_t^2(x, t) dx$$

and

$$P(t) = \frac{T}{3} \int_{-R}^{R} u_x^2(x, t) dx$$

Remark 2.3. *$K(t)$ is usually kinetic energy.*

Remark 2.4. *$P(t)$ is usually potential/elastic energy.*

Theorem 2.1. *Energy Conservation Let*

$$\rho u_{tt} = T u_{xx}$$

Have the properties

1. $u \in C^2(\mathbb{R} \times \mathbb{R})$
2. $u(x, 0) = \phi(x), u_t(x, 0) = \psi(x)$

[11]Check less than or equal for R.

3. $\psi(x) = \phi(x) = 0$ *if* $|x| > R$ *for some* $R \geq 0$

Then we can say that:

$$E(t) = E(0) = \frac{1}{2}\int_{-\infty}^{\infty} \rho|\psi(x)|^2 + T(|\phi_x(x)|^2)dx \tag{2.21}$$

12

Proof. Let $s = R + Ct$ then $\forall\ s \in [0, t]$

$$E(s) = \frac{1}{2}\int_{-R}^{R} \big(\rho(u_t(x,s)\big)^2 + T\big(u_x(x,s)\big)^2 dx$$

We differentiate E with respect to s.

$$\frac{d}{ds}E(s) = \int_{-R}^{R} \rho u_t u_{tt} + T u_x u_{xt} dx \tag{2.22}$$

If we say that $\rho u_{tt} = T u_{xx}$ then equation 2.22 is equivalent to

$$T\int_{-R}^{R} u_t u_{xx} + u_x u_{xt} dx \tag{2.23}$$

Then we can integrate equation 2.23 by parts to give

$$T\int_{-R}^{R} u_{xt}u_x - u_{xx}u_t dx \underbrace{+T\big(u_t + u_x(R,S) - u_t u_x(-R,S)\big)}_{=0}$$

□

13

Lecture 7: 22/1/07

3 The diffusion equation

Where k is the diffusion coefficient, the diffusion equation is:

$$u_t = k u_{xx} \quad k > 0 \tag{3.1}$$

In many cases eqn 3.1 is used to solve heat related problems, like travelling up an iron rod, and the equation is known as the heat equation. (k being the heat conductivity). $u(t, x)$ is either the concentration or the temperature of the substance of interest.

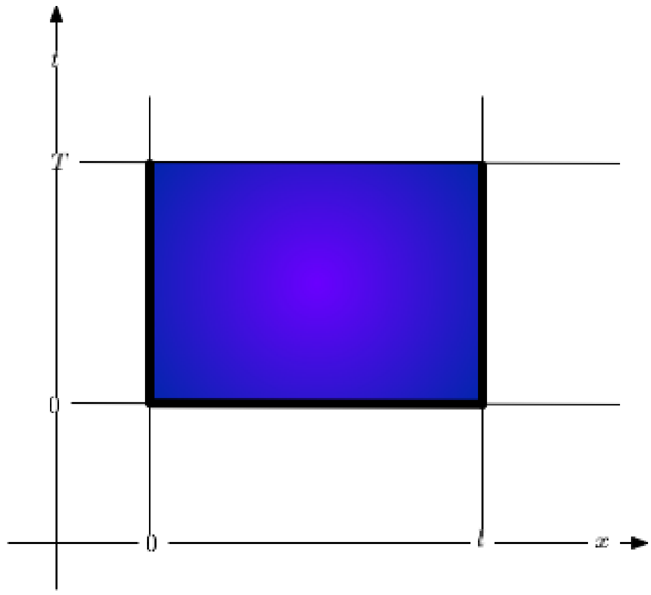

Figure 10: Box with 3 sides where u achieves it's maximum.

3.1 The Maximum Principals

Theorem 3.1. *let* $u \in C^2\left([0,l] \times [0,T]\right)$ *be a solution of equation 3.1. Then* u *assumes its maximum on the set* $(x,t) \in ([0,l] \times [0,T])$, $t = 0$, $x = 0$, $x = l$

Remark 3.1. *Solutions of the wave equation do not satisfy such an estimate. e.g take*

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = 0$$

Outline of the Proof of theorem 3.1 Assume not then there exists $[x_0,t_0] \in [0,l] \times [0,T]$ s.t. $u(x_0,t_0) \geq u(x,t)\ \forall x,t \in [0,l] \times [0,T]$. Since u is differentiable we know that $u_{xx}(x_0,t_0) = 0$ and furthermore $u_{xx}(x_0,t_0) \leq 0$ If we knew that $u_{xx}(x_0,t_0) < 0$ (which we don't) then the equation $u_t = ku_{xx} \Rightarrow u_t < 0$ since $k > 0$. But this contradicts the assumption that $u(x_0,t_0)$ is the maximum of u even if $t_0 = T$. Since we don't know that $u_{xx}(x_0,t_0) < 0$ we have to refine the argument.

Proof of theorem 3.1. Assume M is the maximum value of u on either of the sides of $x = 0$ or $x = l$ or $t = 0$ (as shown in figure 10) Recall from Analysis 2:

Lemma 3.2. *Every continuous function assumes it's maximum on closed bounded intervals.*

We want to show that $u(x,t) \leq M\ \forall(x,t) \subset [0,l] \times [0,T]$

[12]this may be an amazing result, but what it's supposed to be god only knows
[13]This proof makes little sense

lecture 8: 23/1/07 let $\epsilon > 0$ and let $v(x,t) = u(x,t) + \epsilon x^2$. We will show that $v(x,t) \leq M + \epsilon l^2$

$$\forall (x,t) \in [0,l] \times [0,T]$$

From the definition of v it follows that:

$$v(x,t) \leq M + \epsilon l^2$$

If $t = 0$ or $x = 0$ or $x = l$ Furthermore

$$v_t - kv_{xx} = u_t - k(u + \epsilon x^2)_{xx}$$
$$u_t - ku_{xx} - 2\epsilon k = 2\epsilon k < 0$$

Since u solves equation 3.1.

Using the previous argument we conclude that v does not assume it's maximum at a point $(x_0', t_0') \in \big[(0,l) \times (0,T)\big]$ If this were the case. Then $v_x(x_0', t_0') = 0$ and $v_{xx}(x_0', t_0') \leq 0$ as before.

Furthermore, as $v(x_0', t_0')$ is bigger or equal to $v(x_0', t_0 - \delta)$ we have that $v(x_0', t_0') = \lim_{\delta \to 0} \frac{1}{\delta}(v(x_0, t_0) - v(x_0', t_0' - \delta))$

But this is a contradiction to the previously established inequality $v_t - kv_x x < 0$ Hence v doesn't assume it's maximum. $(0,l) \times (0,T)$. Since v is continuous it assumes it's maximum somewhere in $[0,l] \times [0,T]$. This implies that $v(x,t)$ assumes it's maximum if either $x = 0, x = lort = 0$.

Therefore $v(x,t) \leq M + \epsilon l^2$. $(\forall (x,t) \in (0,l) \times (0,T))$. This implies that

$$u(x,t) \leq M + \epsilon(l^2 - x^2) \leq M + \epsilon l^2$$

. Since this is true for any $\epsilon > 0$ we conclude that $u(x,t) \leq M$

$$\forall (x,t) \in (0,l) \times (0,T)$$

□

An application of the maximum principle theorem shows that solutions of the diffusion equation (3.1) are unique if we assume that u satisfies the initial and boundary conditions of the problem. These are:

$$u(x,0) = \phi(x)$$
$$u(0,t) = g(t), \;\; u(l,t) = h(t)$$

Theorem 3.3. *Uniqueness Let $u \in C^2([0,l] \times [0,T])$ be a solution of the diffusion equation $u_t = ku_{xx}$ (equation 3.1) and satisfy the Initial Boundary Conditions (IBC) then u is unique.*

Proof. Let

$$u^{(1)}, u^{(2)} \subseteq C^2([0,l] \times [0,T])$$

be two solutions of 3.1 which satisfy (IBC) with the same functions for ϕ, g and h. Let

$$v(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t) \tag{3.2}$$

Since equation 3.1 is linear, equation 3.2 satisfies equation 3.1 as well. Furthermore we know that:

$$v(x,0) = v(0,t) = v(l,t) = 0$$

The maximum principle implies that $v(x,t) \leq 0 \; \forall (x,t) \in (0,l) \times (0,T)$ The minimum principle implies that $v(x,t) \geq 0$ hence $v(x,t) = 0 \; \forall x,t$. □

Lecture 9: 25/1/07

Theorem 3.4. *Let $u^{(1)},\ u^{(2)} \in C^2([0,l]\times[0,\infty])$ be two solutions of the inhomogeneous heat equation.*

$$u_t = bu_{xx} + f(x,t) \tag{3.3}$$

(f is often called the heat source) If u_1 and u_2 satisfy the initial conditions:

$$u_1(x,0) = \phi_1(x),\ u_2(x,0) = \phi_2(x) \tag{3.4}$$

and the boundary conditions:

$$u_1(0,t) = u_2(0,t) = g(t),\ u_1(l,t) = u_2(l,t) = h(t) \tag{3.5}$$

then this leads to the equation:

$$\int_0^l \big(u_1(x,t) - u_2(x,t)\big)^2 dx \le \int_0^l \big(\phi_1(x) - \phi_2(x)\big)^2 dx \tag{3.6}$$

Remark 3.2. *Inequalities like equation 3.6 which involve integrals of the solutions of PDE's are also frequently called energy estimates. It is preferable to have pointwise estimates such as the maximum principle; however in most practical cases the maximum principle unfortunately doesn't hold so you can't use it. On the other hand it is often not difficult to obtain energy estimates.*

Proof of theorem 3.4. Let $v(x,t) = u_1(x,t) - u_2(x,t)$ then v satisfies

1. $v_t = kv_{xx}$
2. $v(x,0) = \phi_1(x) - \phi_2(x)$
3. $v(0,t) = v(l,t) = 0$

If we start with

$$\int_0^l v^2(x,t)dx \tag{3.7}$$

Then if we differentiate equation 3.7 with respect to t we get:

$$\frac{\partial}{\partial t}\int_0^l v^2(x,t)dx$$

$$= 2\int_0^l v(x,t)v_t(x,t)dx$$

$$= 2\int_0^l v(x,t)ku_{xx}(x,t)dx$$

Then if we integrate this by parts we then get:

$$= -2k\int_0^l v_x^2(x,t)dx + \underbrace{v(0,t)v_x(0,t) - v(l,t)v_x(l,t)}_{=0 \text{ by the boundary conditions}}$$

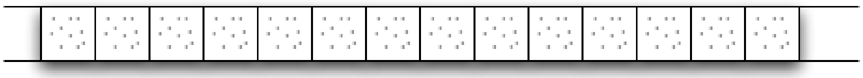

Figure 11: Boxes with particles to simulate diffusion

$$= -2\,k \underbrace{\int_0^l v_x^2(x,t)dx}_{\geq 0} \leq 0$$

Hence

$$\int_0^l v^2(x,t)dx \leq \int_0^l v^2(x,0)dx = \int_0^l (\phi_1(x) - \phi(t))^2 dx$$

□

3.2 Derivation of the diffusion equation

Accurate mathematical models of diffusion are pretty much always non-trivial. This means that to enable them to be covered in this course we will have to work with a simplified system. You can see how this looks in practice in figure 11.

Example 3.1. Assume that $n(i,j) \in \mathbb{N}$ is the number of particles in base i at time j. Both space and time are discrete. Assume that as time advances from j to $j+1$ each particle jumps to one of the two neighbouring boxes (with equal probability). We simplify this model by assuming that

$$n(i,j+1) - n(i,j) = \frac{1}{2}(n(i-1,j) - 2n(i,j) + n(i+1,j))$$

Now if we assume that $n(i,j) = u(hi, h^2 j)$ then setting $x = hi$, $t = h^2 j$ we obtain

$$\frac{1}{h^2}(u(x,t+h^2) - u(x,t)) = \frac{1}{2h^2}u(x+h,t) - u(x,t) + u(x-h,t) \tag{3.8}$$

Then Taylor's theorem implies that:

$$u(x,t+h^2) = u(x,t) + h^2 u_t(x,t) + o(h^2) \tag{3.9}$$

$$u(x+h,t) = u(x,t) + hu_x(x,t) + \frac{1}{2}h^2 u_{xx}(x,t) + O(h^2) \tag{3.10}$$

This implies that:

$$u_t = \frac{1}{2}u_{xx} + O(h)$$

Then as $h \to 0$ we are left with the diffusion equation

$$u_t = \frac{1}{2}u_{xx}$$

with the diffusion coefficient $k = \frac{1}{2}$.

3.3 The diffusion equation on the real line

We want to find functions:

$$u \in C^2(\mathbb{R} \times [0, \infty))$$

which satisfy the diffusion equation $\forall x \in \mathbb{R}$.

Lecture 10: 29/1/07 Before doing this we need to recall several fundamental properties of equation 3.1. If u solves the diffusion equation $u \in C^2(\mathbb{R} \times (0, \infty))$ then the following also solve the diffusion equation:

1. Translations $u(x - y, t)$.
2. Derivatives u_x, u_t, u_{xx} etc.
3. Linear combinations $c_1 u + c_2 v$. (if v solves equation 3.1 as well)
4. Integrals such as $\int_0^x u(s, t)ds$
5. The dilated functions $f_a(x, t) = u(\sqrt{a}x, at)\ \forall a > 0$

Proof. Property 5

$$\frac{\partial}{\partial t}\frac{\partial}{\partial t}v = au_t(\sqrt{a}x, at) \quad \frac{\partial^2}{\partial x^2}v = au_{xx}(\sqrt{a}x, at)$$

A consequence of equation 3.1 is that

$$v(x, t) = \int_{-\infty}^{\infty} u(x - y, t)\phi(y)dy$$

solves equation 3.1 (respective of ϕ) □

3.3.1 Step 1

We look for solutions $Q(x, t)$ of the form:

$$Q(x, t) = y(p) \text{ where } p = \frac{x}{4kt} \tag{3.11}$$

The reason why we want Q to have this special form is that because of property 5 the dilates are solutions here if Q is given by equation 3.11, then the evolution is equivalent dilation. Such solutios are important to understand. The essential properties of equation 3.1.

3.3.2 Step 2

Put $g(p)$ into equation 3.1, this yields an ODE for g.

$$Q_t = g'(p)p_t = {}^{14} = \frac{x}{\sqrt{4kt}}g' \tag{3.12}$$

[14] $\frac{1}{2}\frac{x}{2t} = ????$

$$
\begin{aligned}
Q_x = g'(p)p_x = \frac{1}{4kt}g' \\
Q_{xx} = \frac{1}{4kt}g'' \\
0 = Q_t - kQ_{xx} = \frac{1}{l}\left(-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right)
\end{aligned}
\tag{3.13}
$$

Thus this means that:

$$g'' + 2pg' = 0$$

The complete solution of this ODE is:

$$g(p) = c_1 \int_0^p s^2 s + c_2$$

3.3.3 Step 3

Determine c_1 and c_2 by considering the behaviour of $Q(x,t)$ as $t \to 0$. If $x > 0$ we have that

$$\lim_{t\to 0^+} = \lim_{t\to 0^+} c_1 \int^{\frac{x}{4kt}} e^{-p^2} dp = c_1 \int_0^{\infty} e^{p^2} dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2$$

If $x < 0$ we obtain:

$$\lim_{t\to 0^+} Q(x,t) = -c_1 \frac{\sqrt{\pi}}{2} + c_2$$

The simplest function of a jump is.

$$H(x) = \begin{cases} 0 & x < 0 \\ 1x > 0 \\ \frac{1}{2}x = 0 \end{cases}$$

To obtain this limit as $t \to 0^+$ we have to choose c_1, c_2 in such a way that:

$$c_1 \frac{\sqrt{\pi}}{2} + c_2 \cdot 1,\ c_1 \frac{\pi}{2} - c_2 = 0 \Rightarrow c_1 = \frac{1}{\sqrt{\pi}},\ c_2 = \frac{1}{2}$$

and

$$Q(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-p^2} dp$$

We now define

$$s(x,t) = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}} e^{\frac{-x^2}{4kt}}$$

By property 2 S is also a solution of equation 3.1 given a function ϕ we define by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$

The function S is often called the fundamental solution, this is shown in figure 13. Recall that $u(x,t)$ is also a solution of equation 3.1.

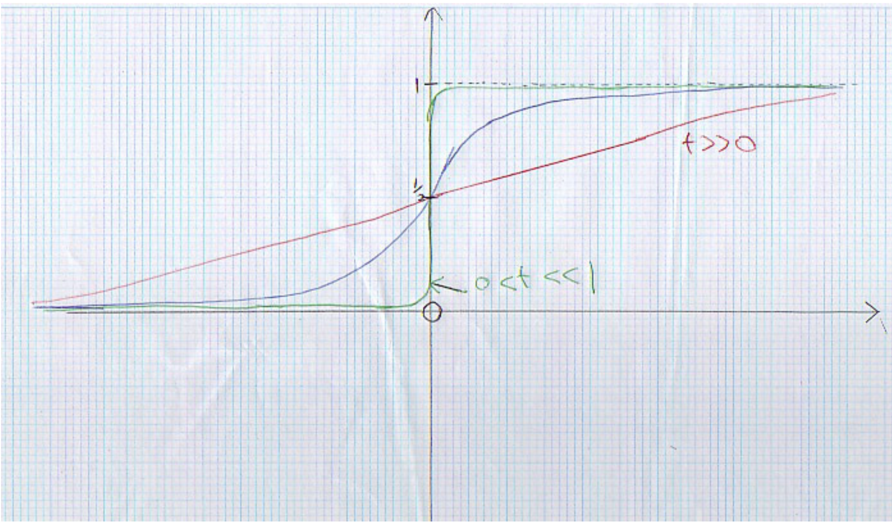

Figure 12: Graph showing u(x,t) and x

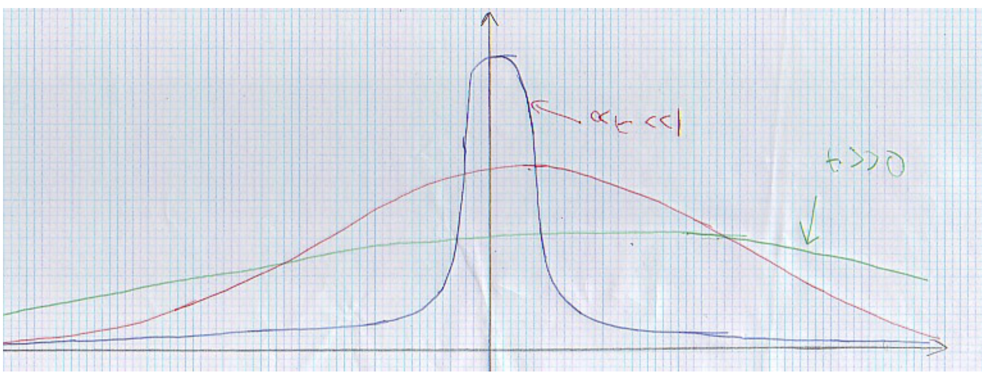

Figure 13: Graph showing S(x,t) and x

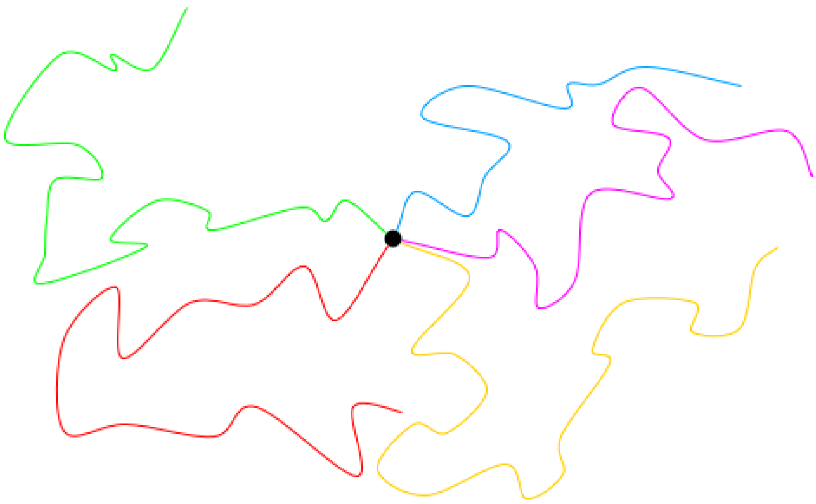

Figure 14: Graph showing brownian motion paths.

Lecture 11:30/1/07

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}$$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$

We expect that $\int_{-\infty}^{\infty} S(x,t)dx$ is constant since $u_t = ku_{xx}$ models the diffusion of a substance. Indeed:

$$\int_{-\infty}^{\infty} S(x,t)dx = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{4kt}} dx$$

If $y = \frac{x}{\sqrt{4kt}}$

$$\frac{1}{\sqrt{\pi}} e^{-y^2} dy = 1$$

Physical interpretation of S: Add a droplet of dye into a glass of water. At $t = 0$ the concentration is high at $x = 0$ and zero everywhere else. After some time the colour reaches all parts of the container. This is well described by $S(x,t)$. At a more microscopic level one can observe small particles which performed "Brownian motion". The paths are shown in figure 14 and the spread of particles at a time t is shown in figure 15. We have to analyse $u(x,t)$ as $t \to 0$. Expect

$$\lim_{t\to 0} u(x,t) = \phi(x)$$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial q}{\partial x}(x-y,t)\phi(y)dy$$

$$= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}(Q(x-y,t))\phi(y)dy$$

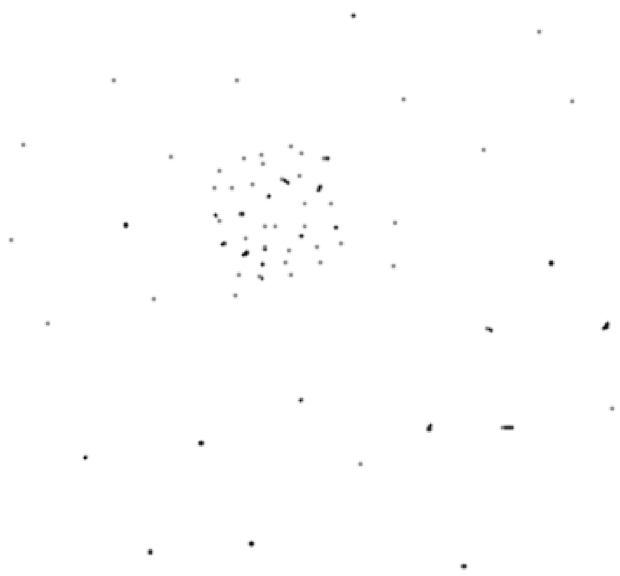

Figure 15: Graph showing brownian motion particles at a time t.

$$= + \int_{-\infty}^{\infty} Q(x-y.t)\phi'(y)dy - \Big[Q(x-y,t)\phi(y)\Big]_{y=-\infty}^{y=\infty}$$

We can drop the second term if we assume that $\lim_{|x|\to\infty} |\phi(x)| = 0$. Under this assumption one obtains.

$$\lim_{t\to 0} u_x(x,t) = \lim_{t\to 0} \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy \tag{3.14}$$

Now since $\lim_{t\to 0^+} Q(x,t) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$ equation 3.14 is equal to:

$$(3.14 = \int_{-\infty}^{x} \phi'(y)dy = \phi(y)\Big|_{y=-\infty}^{y=x} = \phi(x)$$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$

Example 3.2.

$$\phi(x) = e^{-x}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} e^{-y} dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp(-\frac{x^2 - 2xy + y^2 + 4kty}{4kt})dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp(\frac{-(y+2kt-x)^2}{4kt} + kt - x)dy$$

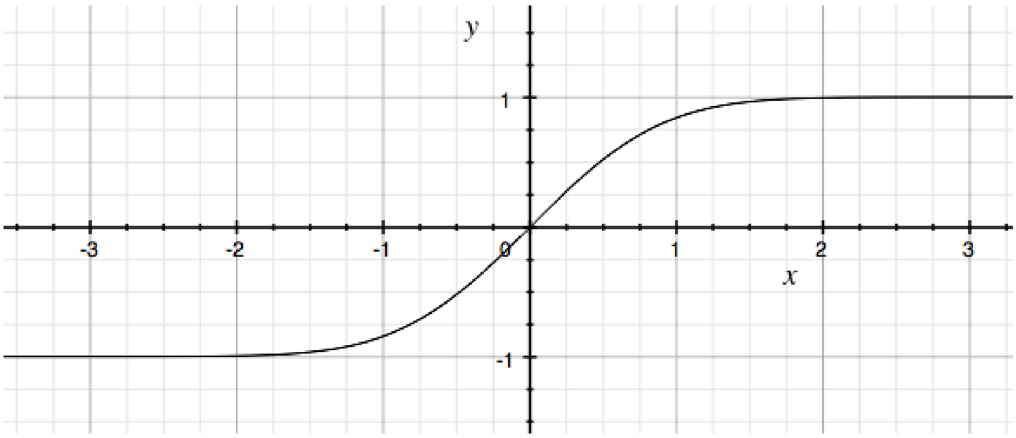

Figure 16: Graph of the function $\text{erf}(x)$

So if we take $p = \frac{1}{4kt}(y + 2kt - x)$

$$= e^{kt-x} \underbrace{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp}_{=1} = e^{kt-x}$$

Example 3.3. Let

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

There is a graph of this in figure 16.

$$Q(x,t) = \frac{1}{2}\frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

Comparison between the wave equation and the diffusion equation.

Property	Diffusion	Wave
Speed of propagation	Infinite	Finite ($\leq$ C)
Singularities at $t > 0$	Immediately Lost	Travel along characteristic
Solutions for $t > 0$	Immediately lost	Travel along characteristic
Solutions exist for $t > 0$	YES	YES
Solutions exist for $t < 0$	NO	YES
Max principle	YES	NO
"Information" lost	YES (gradual) loss of information	NO transported

4 Introduction to Fourier Analysis

4.1 A note on Fourier transforms

Our lecturer has notated Fourier series in a non-standard way:

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(x)$$

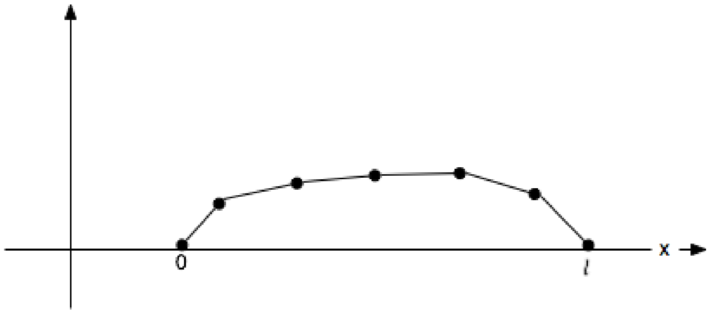

Figure 17: Graph showing chain with Dirichlet boundary conditions

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)$$

Whereas the standard notation is:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \tilde{f}(k)$$

You need to remember this if you look up Fourier transforms on other sites such as wikipedia or Mathworld.

Lecture 12: 1/2/07

4.2 Motivation

Boundary value problems

Separation of variables.

4.3 Boundary Values

This section is also applied in the chapter on Laplace 5

4.3.1 Dirichlet boundary conditions

$$u(0,t) = a, \; u(l,t) = b \text{ if } u \in C([0,l] \times [0,T])$$

Why are such boundary conditions relevant? In the case of the wave equation. Dirichlet boundary conditions model the assumption that the particles at the ends of the chain are "clamped"

4.3.2 Neumann boundary conditions

$$u_x(0,t) = a,\ u_x(l,t) = b$$

Neumann boundary conditions model the assumption that we are pulling with a constant force $f_0 = \frac{a}{T}$, and $f_i = \frac{b}{T}$ on the particles at the end of the chain.

$$\left(c = \sqrt{\frac{T}{\rho}}\right)$$

In the case of the diffusion equation. Dirichlet boundary conditions express the assumption that the concentration (or the temperature) at the boundary is known. (for example when one boils an egg) Neumann boundary conditions model the assumption that we know the flux through the boundary.

4.4 Separation of variables

Example 4.1. Consider again the wave equation

$$u_{tt} = c^2 u_{xx} \tag{4.1}$$

with Dirichlet boundary conditions $u(0,t) = u(l,t) = 0$.

Can we find solutions of the wave equation of example 4.1?

As originally done by Euler, we fix special initial conditions:

$$u(x,0) = a_k \sin\left(\frac{k\pi x}{l}\right),\ u_t(x,0) = b_k \sin\left(\frac{k\pi x}{l}\right) \tag{4.2}$$

We plug this Ansatz (assuming the solution can be written in a certain way [1, 2]) into the wave equation (4.1) and obtain:

$$v''(t)\sin\left(\frac{k\pi x}{l}\right)$$
$$= -c^2 v(t)\frac{k^2\pi^2}{l^2}\sin\left(\frac{k\pi x}{l}\right)$$
$$v'' = -\frac{c^2 k\pi}{l^2} v$$

then v solves the wave equation. The complete solution of this ODE is:

$$v(t) = \alpha_k \cos\left(\frac{ck\pi}{l}t\right) + \beta_k \sin\left(\frac{ck\pi}{l}t\right)$$

$u(x,t)$ satisfies the Dirichlet boundary conditions since:

$$\sin(0) = \sin(k\pi) = 0\ \forall k \in \mathbb{Z}$$

The constants α_k and β_k are determined by the initial conditions mentioned in equation 4.2. This implies that $\alpha_k = a_k$ and $\beta_k = \frac{l}{ck\pi} b_n$. Hence

$$u(x,t) = \left(a_k \cos\left(\frac{ck\pi}{l}t\right) + \frac{l}{ck\pi} b_k \sin\left(\frac{ck\pi}{l}\right)\right.$$

$\sin\left(\frac{\pi k x}{l}\right)$ is a solution of the initial boundary value problem.

4.4.1 By linearity

If

$$u(x,0) = \sum_{k=1}^{n} a_k \sin\left(\frac{k\pi x}{l}\right) = \phi(x)$$

$$u_t(x,0) = \sum_{k=1}^{n} b_k \sin\left(\frac{k\pi x}{l}\right) = \psi(x)$$

then:

$$u(x,t) = \sum_{k=1}^{n} a_k \cos\left(\frac{ck\pi}{l}t\right) + \frac{b_k l}{ck\pi} \sin\left(\frac{ck\pi}{l}t\right) \sin\left(\frac{k\pi x}{l}\right)$$

[15] Solves the initial boundary value points $u_{tt} = c^2 u_{xx}$

$$u(0,t) = u(l,t) = 0$$

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x)$$

Euler had in 1749 the idea that this calculation can explain the musical notes. The lowest frequency we can generate with our approach is $\frac{\pi c}{l}$, the next is $\frac{2\pi c}{l}$ etc. etc. In particular the frequency does not depend on $a_1, a_2, \ldots$. The frequency is proportional to c and anti-proportional to l, i.e. we halve the length and we double the frequency.

Lecture 13: 5/2/07

$$\phi(x) = \sum a_k \sin(kx)$$

$$\psi(x) = \sum b_k \sin(kx)$$

4.5 Fourier co-efficients

We will consider the following questions.

1. Let $f \in C(\mathbb{R})$, can we find a_k, b_k s.t $f(x) = \sum_{k=0}^{\infty}(a_k \cos(kx) + b_k \sin(kx))$
2. Which type of convergence can we expect in (1)? How fast does the series converge?

Before we start we simplify the notation by introducing complex numbers. $z = a + ib$, $a, b \in \mathbb{R}$ $z = re^{i\theta}$ $r \geq 0$ is the modulus, $\theta \in [0, 2\pi)$ is the argument.

Since

$$\mathrm{Re}((a+ib)e^{ikx}) = a\cos(kx) + b\sin(kx)$$

We see that:

$$\sum_{k=0}^{\infty} a_k \cos(kx) + b_k \sin(kx) = f(x)$$

$$= \mathrm{Re}\left(\sum_{k=0}^{\infty}(a_k - ib_k)e^{ikx}\right)$$

[15] check this

It is more convenient to work with complex valued functions. $f : \mathbb{R} \to \mathbb{C}$ and sum over $k \in \mathbb{Z}$ i.e.

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

$\hat{f}(k) \in \mathbb{C}$[16] are denoted by <u>Fourier coefficients</u>. We obtain real-valued functions if $\hat{f}(-k) = \overline{\hat{f}(h)}$

$$\overline{f(x)} = \sum_{k=-\infty}^{\infty} \overline{\hat{f}(k)} e^{-ikx} = \sum_{k=-\infty}^{\infty} \hat{f}(-k) e^{-ikx} \underbrace{=}_{k'=-k} \sum_{k'=-\infty}^{\infty} \hat{f}(k') e^{ik'x} = f(x)$$

For each $n \in \mathbb{N}$ the n'th partial Fourier series is given by

$$f_n(x) = \sum_{k=-n}^{n} \hat{f} e^{ikx}$$

Note that f_n is $2\pi -$ periodic

$$f_n(x + 2\pi) = f_n(x)$$

this implies that $f_n(x)$ will not converge to $f(x)$ for $x \in \mathbb{R}$ if f is not $2\pi -$ periodic.

Q: Given a $2\pi -$ periodic function f, how can you find $\hat{f}(k)$

Preliminary calculation. Let $k' = -k$

$$\int_0^{2\pi} e^{ikx} e^{ik'x} dx = \int_0^{2\pi} e^{i(k+k')x} dx$$

$$= \int_0^{2\pi} e^0 dx = 2\pi$$

Now let $k' \neq -k$ then:

$$\int_0^{2\pi} e^{ikx} e^{ik'x} dx = \int_0^{2\pi} e^{i(k+k')x} dx$$

$$= \left[\frac{1}{i(k+k')} e^{i(k+k')x} \right]_{x=0}^{x=2\pi} = -\frac{1}{i(k+k')} + \frac{e^{i(k+k')2\pi}}{i(k+k')}$$

$$= \frac{1}{i(k+k')} \left(\underbrace{\cos((k+k')2\pi)}_{=1} + i \underbrace{\sin((k+k')2\pi)}_{=0} - 1 \right)$$

$$= \frac{1}{i(k+k')} \times 0 = 0$$

We could have calculated this using sin and cos [17] instead of e^{ikx} since $e^{ikx} = \cos kx + i \sin kx$. Now we can compute $\hat{f}(k)$. Assume

$$f(x) = \sum_{k \in \mathbb{Z}} e^{ikx} \hat{f}(k)$$

[16] Note that if you are doing *real* modules (such as Mathematical Methods for Physicists ;)) $\tilde{f}$ is used instead of $\hat{f}$. Also remember that because I am those modules too I might get my notation confused. Finally remember that $\hat{f}(k) = \frac{1}{2\pi} \cdots f(x)$ and $f(x) = \cdots \hat{f}(k)$ rather than multiplying each by $\frac{1}{\sqrt{2\pi}}$

[17] Sorry to bring it up again, but this is done in more detail Mathematical methods for physicists, lecture 1 (lecture 2 is the complex case) (see it's not a joke course after all!) the link is `http://www.warwick.ac.uk/go/px261` Warwick Login required.

Choose $k' \in \mathbb{Z}$ multiply the equation above with

$$e^{ik'x} \Rightarrow f(x)e^{ikx} = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{i(k+k')x}$$

$$\int_0^{2\pi} f(x)e^{ik'x}dx = \int_0^{2\pi} \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{(k+k')x}dx$$

Now we swap integral and summation. We will not give mathematical justification for this step.[18]

$$= \sum_{k\in\mathbb{Z}} \underbrace{\int_0^{2\pi} \hat{f}(k)e^{(k+k')x}dx}_{0 \text{ if } k'\neq -k \text{ and } 1 \text{ if } k'=-k}$$

$$\Leftrightarrow \int_0^{2\pi} f(x)e^{-kx}dx = 2\pi\hat{f}(k)$$

$$\Leftrightarrow \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx = \hat{f}(k)$$

Lecture 14: 6/2/07

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}f(x)dx$$

$$f(x) = \sum_{k=-\infty}^{\infty} e^{ikx}\hat{f}(k)$$

Example 4.2.

$$f(x) = \sin x$$

Clearly f is $2\pi - \textit{periodic}$.

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} (\cos kx - i\sin kx)\sin x\ dx$$

$$\frac{1}{2\pi}\underbrace{\int_0^{2\pi} \sin x\cos kx\ dx}_{=0} - \frac{i}{2\pi}\int_0^{2\pi} \sin x \sin kx\ dx$$

$$= \frac{i}{2\pi}\int_0^{2\pi} \sin x\sin kx = \begin{cases} -\frac{i}{2} & k=1 \\ \frac{i}{2} & k=-1 \\ 0 & k\in\mathbb{Z}\backslash\{-1,1\} \end{cases}$$

Compute the Fourier series:

$$\hat{f}(-1)e^{-ix} + \hat{f}(1)e^{ix}$$

$$-\frac{i}{2}\Bigg(\cos(-x) + i\sin x\Bigg) + \Bigg(\cos x + i\sin x\Bigg) = \sin x$$

[18]:rolleyes: it's not Physics ;)

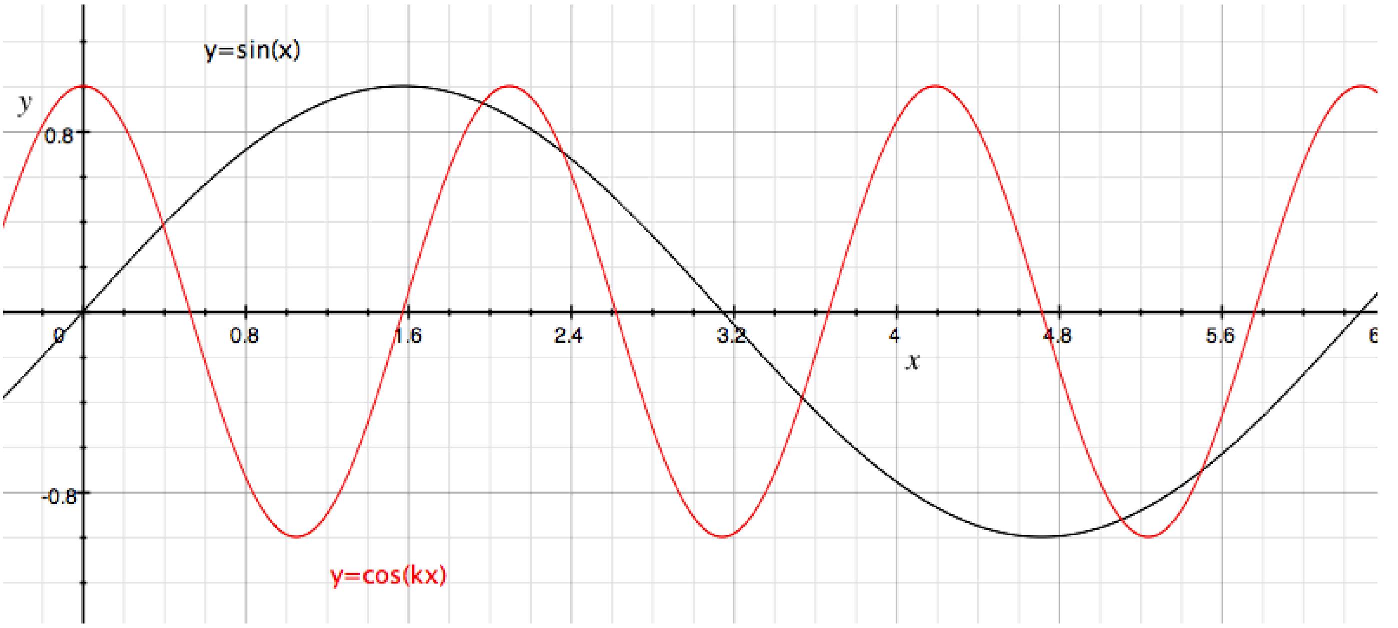

Figure 18: Graph showing $\sin x$ and $\cos kx$

Example 4.3.

$$f(x) = \begin{cases} x & x \in [0, 2\pi] \\ x - 2\pi k & \text{if } k \in \mathbb{Z} \text{ s.t } x - 2\pi k \in [0, 2\pi) \end{cases}$$

Lets now compute $\hat{f}(k)$.

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} x dx = \frac{1}{2\pi} [x^2]_{x=0}^{x=2\pi}$$

Let $k \subset \mathbb{Z} \backslash \{0\}$ then:

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} x e^{-ikx} dx$$

$$= \frac{-1}{2\pi ik} \underbrace{\int_0^{2\pi} e^{-ikx} dx}_{=0} - \left[\frac{1}{2\pi ik} x e^{-ikx} \right]_{x=0}^{x=2\pi} = \frac{-1}{\cancel{2\pi} ik} \cancel{2\pi} e^{-ik2\pi} = \frac{-1}{ik}$$

4.6 Partial Fourier Series

$$f_n(x) = \pi + \sum_{k=-n}^{n} \left(\frac{-1}{ik} \right) e^{ikx} = \pi + \sum_{k=1}^{n} \cancel{-\frac{1}{ik} \cos kx + \frac{1}{ik} \cos(-kx)} + \sum_{k=-n}^{n} \frac{-}{1} \cancel{i} k \cancel{i} \sin kx + \frac{1}{\cancel{i} k} \cancel{i} \sin(-kx)$$

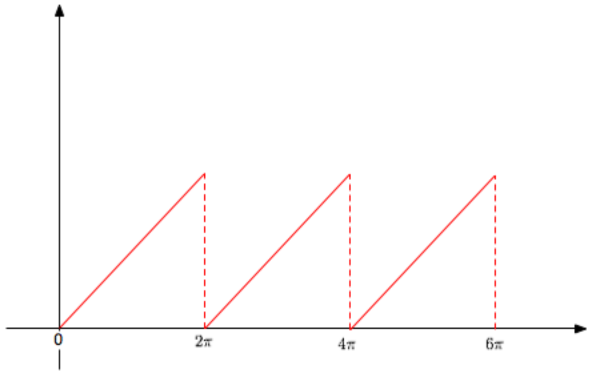

Figure 19: Graph showing the "sawtooth function" ($y = x$ periodic over 2π)

4.7 Convergence of Fourier Series

Theorem 4.1. *Let $f \in C([0, 2\pi], \mathbb{C})$. Among all possible choices of $2n + 1$ constants, $c_{-n}, \dots, c_n$ the choice that minimises:*

$$\int_0^{2\pi} \left| f(x) - \sum_{k=-n}^{n} e^{ikx} c_k \right|^2 dx$$

is

$$c_k = \hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

Remark 4.1. *In particular c_k does not depend on n!*

Proof.

$$E_n = \int_0^{2\pi} \left| f(x) - \sum_{|k| \le n} e^{ikx} c_k \right|^2 dx$$

$$\int_0^{2\pi} \left(f(x) - \sum_{|k| \le n} e^{ikx} c_k \right) \left(\overline{f(x)} - \sum_{|k| \le n} e^{-ikx} \overline{c_k} \right) dx$$

$$= \int_0^{2\pi} |f(x)|^2 dx - \sum_{|k| \le n} \overline{c_k} f(x) e^{-ikx} - c_k f(x) e^{ikx} dx + \sum_{|k| \le n} c_k \overline{c_k} \int_0^{2\pi} e^{i(k-k')x} dx$$

$$= \int_0^{2\pi} |f(x)|^2 dx + 2\pi \sum_{|k| \le n} |c_k|^2 - \sum_{|k| \le n} \int_0^{2\pi} c_k e^{ikx} f(x) + \overline{c_k e^{ikx} f(x)} dx$$

We complete the square

$$E_n = \int_0^{2\pi} |f(x)|^2 dx + 2\pi \sum_{|k| \le n} \left| c_k - \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \right|^2 - 2\pi \sum_{|k| \le n} \left| \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx \right|^2$$

Lecture 15: 8/2/07

$$= \int^{2\pi} |f(x)|^2 dx - 2\pi \sum_{|k|\leq n} |\hat{f}(k)|^2 + 2\pi \sum_{|k|\leq n} \left| c_k - \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx \right|^2$$

The last expression shows that E_n is minimal if:

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx = \hat{f}(k)$$

In particular:

$$\int_0^{2\pi} f(x) - \sum_{|k|\leq n} e^{ikx} |\hat{f}(k)|^2 \quad [19]$$

$$= \int_0^{2\pi} |f(x)|^2 dx - 2\pi \sum_{|k|\leq n} |\hat{f}(k)|^2$$

□

4.8 Consequences of this

1. Bessels inequality

$$2\pi \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2 \leq \int_0^{2\pi} |f(x)|^2 dx$$

2. If

$$\int_0^{2\pi} \left| f(x) - \sum_{k\in\mathbb{Z}} e^{ikx} \hat{f}(k) \right| dx = \lim_{n\to\infty} \int_0^{2\pi} \left| f(x) - \sum_{|k|\leq n} e^{ikx} \hat{f}(k) \right|^2 = 0$$

then

$$\int_0^{2\pi} |f(x)|^2 dx = 2\pi \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2$$

4.9 Parsevals equality

A consequence from Bessel's inequality is the Riemann Lebersgue lemma

Lemma 4.2.

$$\lim_{k\to\infty} \int_0^{2\pi} \sin(kx) f(x) dx = \lim_{k\to\infty} \int_0^{2\pi} \cos(kx) f(x) dx = 0 \quad \forall \text{ functions } f \in \mathrm{C}^0([0, 2\pi], \mathbb{C})$$

(actually we don't really need the continuity of f.)

[19] I don't know what the fuck this line means

Proof. Assume without loss of generality (wlog)that f is a real valued. Otherwise prove the RL lemma for the real and imaginary parts of f separately.

$$\left|\int_0^{2\pi} \sin(kx) f(x) dx\right| = \left|2\pi \ \mathrm{Im}\hat{f}(k)\right| \leq 2\pi |\hat{f}(k)|$$

Since $\sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2 < \infty$ we have that: $\lim_{k\to\infty} |\hat{f}(k)|^2 = 0$. □

Definition 4.1. ***Three notions of convergence***
Let

$$f, \ f_n : [0, 2\pi] \to \mathbb{C}$$

We say that $f_n \to f$:

1. *Pointwise*

$$\lim_{n\to 0} f_n(x) = f(x) \ \forall x \in [0, 2\pi]$$

2. *Uniform*

$$\lim_{n\to\infty, x\in[0,2\pi]} \sup |f_n(x) - f(x)| = 0$$

3. *Mean squared (or L^2)*

$$\lim_{n\to\infty} \int_0^{2\pi} |f_n(x) - f(x)|^2 dx = 0$$

Clearly uniform convergence implies pointwise convergence. Also

$$\int_0^{2\pi} |f_n(x) - f(x)|^2 \leq 2\pi \sup_{x\in[0,2\pi]} |f_n(x) - f(x)|^2$$

Hence uniform convergence implies L^2 convergence. The other possible implications are all false. Homework

Theorem 4.3. *Pointwise convergence of the Fourier series Let $f \in \mathrm{C}^1(\mathbb{R})$ be 2π-periodic. Then*

$$\lim_{n\to\infty} \sum_{|k|\leq n} e^{ikx} \hat{f}(k) \Rightarrow f(x) \quad \forall x \in [0, 2\pi]$$

Proof. Step 1: Find a concise representation of the partial Fourier series.

$$f_n(x) = \sum_{k=-n}^{n} e^{ikx} \hat{f}(k)$$

$$= \sum_{k=-n}^{n} \frac{1}{2\pi} \int_0^{2\pi} e^{ik(x-x')} f(x') dx'$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{2n} e^{i(k-n)(x-x')} f(x') dx'$$

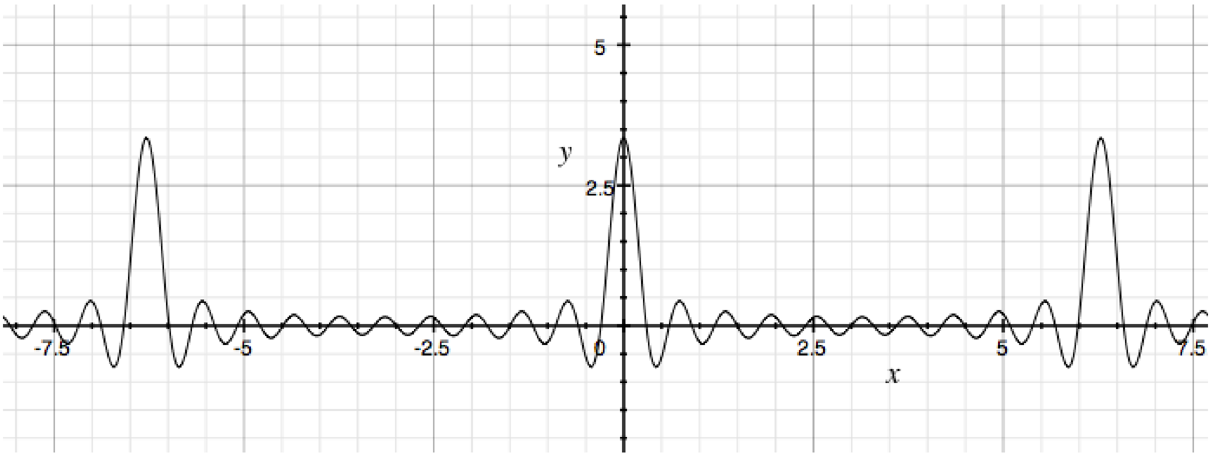

Figure 20: Graph showing the dirichlet kernel for n=10

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-in(x-x')} \sum_{k=0}^{2\pi} \left(e^{i(x-x')} \right)^k f(x')dx'$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \frac{e^{-ni(x-x')} - e^{i(n+1)(x-x')}}{1 - e^{i(x-x')}} f(x')dx'$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \frac{e^{i(n+\frac{1}{2})(x-x')} - e^{i(n+\frac{1}{2})(x-x')}}{e^{-\frac{1}{2}(x-x')} - e^{i(x-x')}} f(x')dx'$$

$$\int_0^{2\pi} \frac{1}{2\pi} \frac{\sin\left((n+\frac{1}{2})(x-x')\right)}{\sin\left(\frac{1}{2}(x-x')\right)} f(x')dx'$$

□

Definition 4.2. *The function*

$$K_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi} \frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\left(\frac{1}{2}x\right)}$$

is the Dirichlet kernel. As we can see in Figure 20 the function reaches a maximum of $2n+1$ *at* $2k\pi$ *(for* $k \in \mathbb{Z}$*) and a minimum of* $\frac{1}{2\pi}$ *at* $(2k+1)\pi$

Lecture 16: 12/2/07

Theorem 4.4.

$$f \in C^2(\mathbb{R}),\ f \text{ is } 2\pi \text{ −periodic}$$

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$$

Then

$$\lim_{n\to\infty} \sum_{|k|\le n} e^{ikx} \hat{f}(k) = f(x)$$

in the pointwise sense

Proof. We have already showed that:

$$f_n(x) = \int_0^{2\pi} K_n(x-x') \,{}^{20}\, f(x')dx$$

where

$$K_n(\theta)_{\text{Dirichlet Kernel}} = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{-ik\theta}$$

It is clear by construction that K_n is 2π periodic and:

$$\int_0^{2\pi} K_n(\theta)d\theta = \frac{1}{2\pi}\sum_{k=-n}^{n} \underbrace{\int_0^{2\pi} e^{-ikx}dx}_{\substack{=0 \quad k\neq 0 \\ 2\pi \quad k=0}} = \frac{1}{\cancel{2\pi}}\cancel{2\pi} = 1$$

$$\int_0^{2\pi} K_n(x-x')f(x')dx = \int_x^{x-2\pi} K_n(\theta)f(x-\theta)d\theta$$

where $\theta = x - x'$

$$\underbrace{=}_{K_n \text{ and } f \text{ are both } 2\pi \text{ periodic}} \int_0^{2\pi} K_n(\theta)f(x-\theta)d\theta$$

□

Now we can analyse $f(x) - f_n(x)$

$$f(x) - f_n(x) = \int_0^{2\pi} K_n(\theta)(f(x) - f(x-\theta))d\theta$$

$$= \int_0^{2\pi} \frac{1}{2\pi}\sin\left(\left(n+\frac{1}{2}\right)\theta\right)\frac{f(x)-f(x-\theta)}{\sin\left(\frac{\theta}{2}\right)}d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(n\theta)\cos\left(\frac{\theta}{2}\right)\frac{f(x)-f(x-\theta)}{\sin\left(\frac{\theta}{2}\right)}d\theta + \frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta)(f(x)-f(x-\theta))d\theta = I_1 + I_2$$

We assume without loss of generality that f is real, then the Fourier coefficient I_1 is the imaginary part of the *nth* Fourier coefficient of the function.

$$g(\theta) = \cos\left(\frac{\theta}{2}\right)\left[\frac{f(x)-f(x-\theta)}{\sin\left(\frac{\theta}{2}\right)}\right]$$

and I_2 is the real part of the *nth* Fourier coefficient of the function

$$g_2(\theta) = f(x) - f(x-\theta)$$

[20] This is similar to the dirac delta function.

Since g_2 is cts the RL lemma implies that

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta)g_2(\theta)d\theta = 0 = \lim_{n\to\infty} I_2$$

Now let

$$\tilde{g}_1(\theta) = \frac{f(x)-f(x-\theta)}{\sin\left(\frac{x}{\theta}\right)}$$

We will now show that $\tilde{g}_1 \in C(\mathbb{R})$. Clearly $\tilde{g}_2 \in C(\mathbb{R}\backslash 2\pi\mathbb{Z})$. Since $\sin\left(\frac{\theta}{2}\right)$ is cts and $\neq 0$ if $\theta \in \mathbb{R}\backslash x + 2\pi\mathbb{Z}$ Let $\theta \in x + 2\pi\mathbb{Z}$ without loss of generality $x = \theta$. We use that and that $\tilde{g}_1$ is cts if and only if it is sequentially cts. Let $\theta_n \in \mathbb{R}\backslash x + 2\pi\mathbb{Z}$ s.t. $\lim_{n\to\infty}\theta_n = 0$ We claim that

$$\lim_{n\to\infty} g_2(\theta_n) = 2\pi f'(x)$$

$$\lim_{n\to\infty}\frac{f(x)-f(x-\theta_n)}{\sin\left(\frac{\theta_n}{2}\right)} \underbrace{=}_{\text{L'Hopital}} \lim_{n\to\infty}\frac{f'(x-\theta_n)}{\frac{1}{2}\cos\left(\frac{\theta_n}{2}\right)} = 2f'(x)$$

The RL lemma implies that

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\cos\left(\frac{\theta}{2}\right)g_1(\theta)\sin(n\theta)d\theta$$

Theorem 4.5. *Let $f \in C^2(\mathbb{R})$, be $2\pi-\text{periodic}$. Then*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|f(x)-f_n(x)| = 0$$

i.e the Fourier series converges uniformly and not only pointwise.

Proof. The proof is based on an important principle in Fourier Analysis. The decay rule of FC (Fourier Coefficient) is controlled by the smoothness of f. More precisely we will show next that

$$|\hat{f}(k)| \le C|k|^{-s} \text{ for } k \in \mathbb{Z}\backslash\{0\}$$

Where $C > 0$ is a constant depending on f but not k.

If $f \in C^s(\mathbb{R})$ and $s \in \mathbb{N}$ and is 2π periodic.

$$|\hat{f}(k)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(x)e^{ikx}dx\right| \underbrace{=}_{\text{parts}} \frac{1}{2\pi}\left|\frac{1}{ik}\int_0^{2\pi} f'(x)e^{-ikx}dx\right|$$

We repeat this s times

$$= \frac{1}{2\pi}\frac{1}{|k|^s}\left|\int_0^{2\pi} f^{(s)}(x)e^{-ikx}dx\right| \le \frac{1}{2\pi}\frac{1}{|k|^s}2\pi\sup_{x\in[0,2\pi]}|f^{(s)}(x)|$$

$C = \sup_{x\in[0,2\pi]}|f^{(s)}(x)|$

Lecture 17: 13/2/07 2π-periodic, then $\exists c > 0$ s.t.

$$|\hat{f}(k)| \leq \frac{C}{|k|^2}$$

$$\forall k \in \mathbb{Z}\backslash\{0\}$$

we have $f \in C^2(\mathbb{R})$ hence $s = 2$

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq \xi_{|k|>\min\{m,n\}} |e^{ikx}\hat{f}(k)| \\ &\leq \xi_{|k|>\min\{m,n\}} |\hat{f}(k)| \\ &\leq 2 \sup_{x\in[0,2\pi]} |f''(x)| \sum_{k=\min\{m,n\}+1}^{\infty} \frac{1}{k^2} \\ &\leq 2 \sup_{x\in[0,2\pi]} |f''(x)| \int_{\min\{m,n\}+1}^{\infty} (k-1)^{-2} dk \\ &= \frac{2}{\min\{n,m\}} \sup_{x\in[0,2\pi]} |f''(x)| \end{aligned} \tag{4.3}$$

We choose $\min\{m, n\}$.

$$\geq \frac{2 \sup_{x\in[0,2\pi]} |f(x)|}{\epsilon}$$

Then

$$|f_n(x) - f_m(x)| \leq \epsilon \ \forall x \in \mathbb{R}$$

$$\frac{\sup_{x\in[0,2\pi]} |f''(x)|}{\epsilon}$$

does not depend on x the sequence $f_n(x)$ converges uniformly. □

$$x + 2\pi\mathbb{Z} \to 2\pi\mathbb{Z}$$

$\tilde{g}$ is possibly discontinuous at $2\pi\mathbb{Z}$.

Theorem 4.6.

$$f \in C^2(\mathbb{R}) \ 2\pi\text{−periodic}$$

Proof. Intermediate result

$$f \in C^s(\mathbb{R})$$

□

Theorem 4.7. *1. $f \in C^s(\mathbb{R})$, 2π-periodic then $\exists C > 0$ s.t $|\hat{f}(k)| \leq \frac{C}{|k|^s}$*

2. Let $\hat{f}(k) \in \mathbb{C}, k \in \mathbb{Z}$ s.t. $|\hat{f}(k)| \leq C|k|^{-r}$ for some $C > 0, r > 1, r \in R$ then

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sum_{k=-n}^{n} e^{ikx}\hat{f}(k)$$

exists and $f \in C^s(\mathbb{R})$

$$s \in (\mathbb{N} \cup \{0\}) \cap [0, r-1)$$

(alternatively $s < r - 1, s \in \mathbb{N} \cup \{0\}$.)

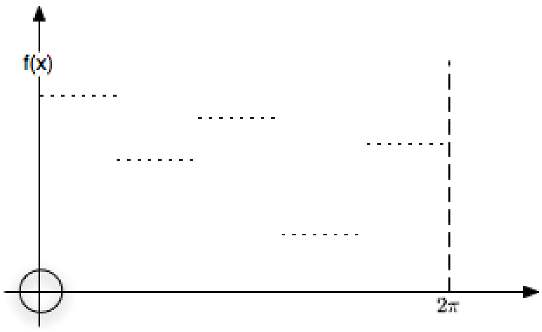

Figure 21: I don't know what this is supposed to be a figure of.

Proof. 1. Has been established in the preceding proof.

2. Analysis II, $f_n \in C^s(\mathbb{R})$ is a sequence of functions. If $f_n^{(s)}$ (sth derviative of f_n.) converges uniformly to some function f then $f \in C^s(\mathbb{R})$

In our case:

$$f_n(x) = \sum_{k=-n}^{n} e^{-ikx} \hat{f}(k)$$

$$f_n^{(s)}(x) = \sum_{k=-n}^{n} (ik)^s e^{ikx} \hat{f}(k)$$

Clearly $f_n^{(s)}$ converges uniformly if:

$$|k|^s |\hat{f}(k)| \leq |k|^{1+\epsilon}$$

□

Then let f be a 2π periodic piecewise cts function. (i.e. $\exists D \in [0, 2\pi]$ s.t. $f \in C(\mathbb{R} \backslash (D + 2\pi\mathbb{Z})))$ If $\forall x_0 \in D$ the left-hand limit

$$\lim_{x \to x_0^+} f(x)$$

The right hand limit

$$\lim_{x \to x_0^-} f(x)$$

exists. Then the partial Fourier series:

$$f_n(x) = \sum_{k=-n}^{n} e^{-ikx} \hat{f}(k)$$

converges pointwise as $n \to \infty$ to

$$\begin{cases} f(x) & \forall x \in \mathbb{R} \backslash (D + 2\pi\mathbb{Z}) \\ \frac{1}{2} \lim_{x \to x_0^-} f(x) + \frac{1}{2} \lim_{x \to x_0^+} f(x) & x \in D + 2\pi\mathbb{Z} \end{cases}$$

Proof Strass p134,135

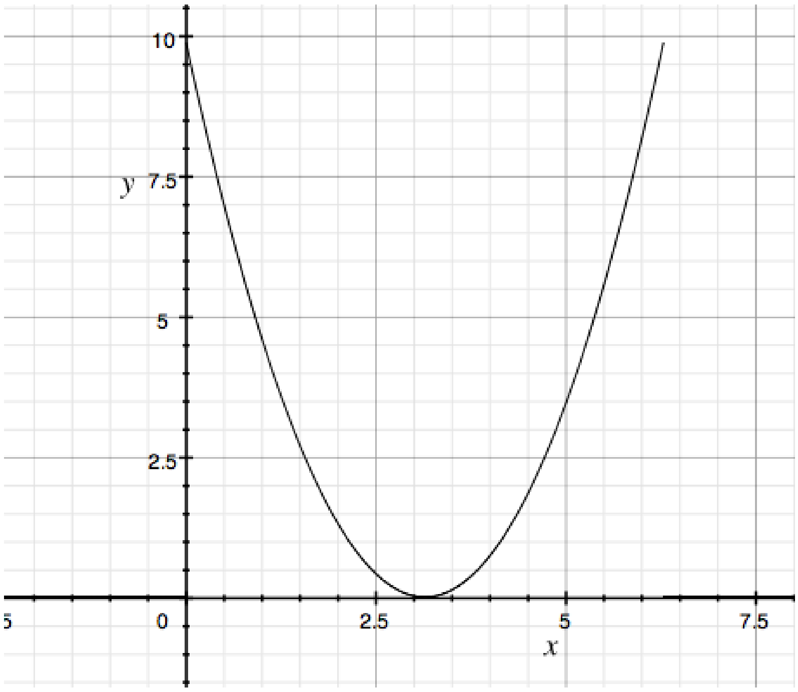

Figure 22: Graph of $((x-\pi)^2$

4.9.1 Applications

Explicit evaluation of sums like:

$$\xi(s) = \sum_{k=1}^{\infty} k^{-2}$$

($\xi(s)$ is the Riemann-Zeta function.)

$$\xi(s) = \sum_{k=1}^{\infty} k^{-s}, \ s \in \mathbb{C}$$

[21] Let:

$$f(x) = (x-\pi)^2$$

If $x \in [0, 2\pi]$ then we can see a graph of this in figure 22.

$$\hat{f}(0) = \frac{\pi^2}{3} \hat{f}(k) = \frac{2}{k^2}$$

if $k \in \mathbb{Z} \backslash \{0\}$ (Homework) Pointwise Convergence of:

$$f_n(x) = \sum_{k=-\infty}^{\infty} e^{-ikx} \hat{f}(k)$$

[21]You can win the Clay prize for finding the zeros (roots) this means you get lots of money if you solve it (possibly 1 millon USD (approximately 1 given the current exchange rate)

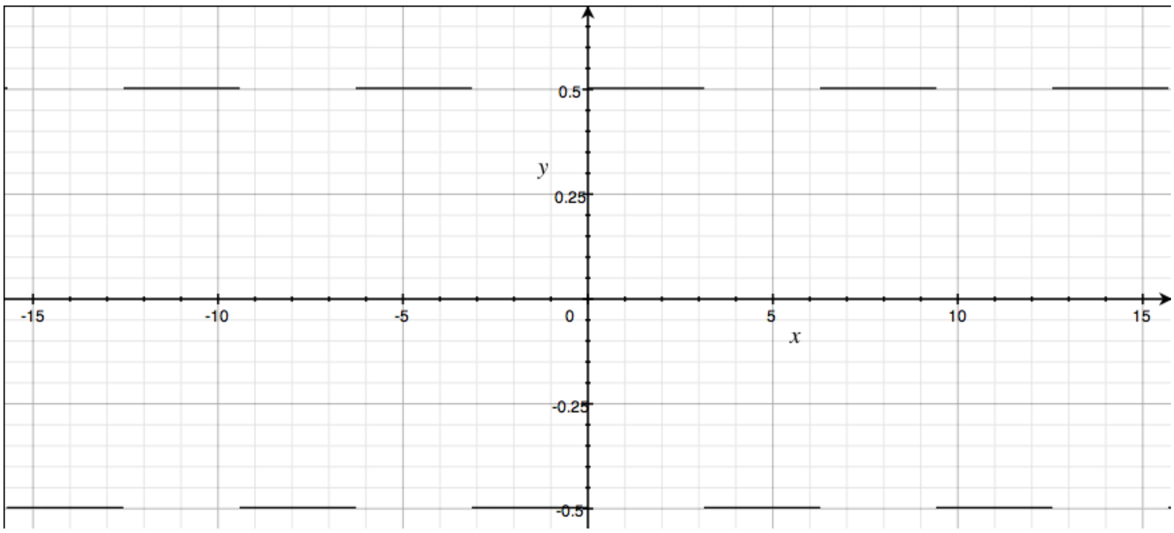

Figure 23: Graph of $f(x)$ periodically extended $\forall x \in \mathbb{R}$

$$= \pi^2 + 4\sum_{k=1}^{n} \frac{1}{k^2}\cos(kx)$$

implies for $x = 0$ that

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{k=1} \frac{1}{k^2}\frac{\pi^2}{6}$$

Lecture 18:15/2/07

4.10 The Gibb's phenomenon

How do Fourier series converge near (jumps) discontinuities? [22]

Example 4.4.

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \pi] \\ -\frac{1}{2} & \text{if } x \in [\pi, 2\pi] \end{cases}$$

and 2π periodically extended $\forall x \in \mathbb{R}$, this is shown in figure 23 Clearly $\hat{f}(0) = 0$, let $k \in \mathbb{Z}\backslash\{0\}$

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$

$$= \frac{1}{4\pi}\int_0^{\pi} e^{-ikx}dx - \frac{1}{4\pi}\int_{\pi}^{2\pi} e^{-ikx}dx$$

$$= \frac{1}{4\pi ik}\left(\left[e^{-ikx}\right]_{x=0}^{x=\pi} - \left[e^{-ikx}\right]_{x=\pi}^{x=2\pi}\right)$$

[22]These mean that the Fourier series converges pointwise but not uniformly, the Gibb's phenomenon are the jumps up near the discontinuities and go up about 0.09 times the size of the discontinuity (not mathematically rigourous) as we covered in my favourite module Mathematical Methods for Physicists.

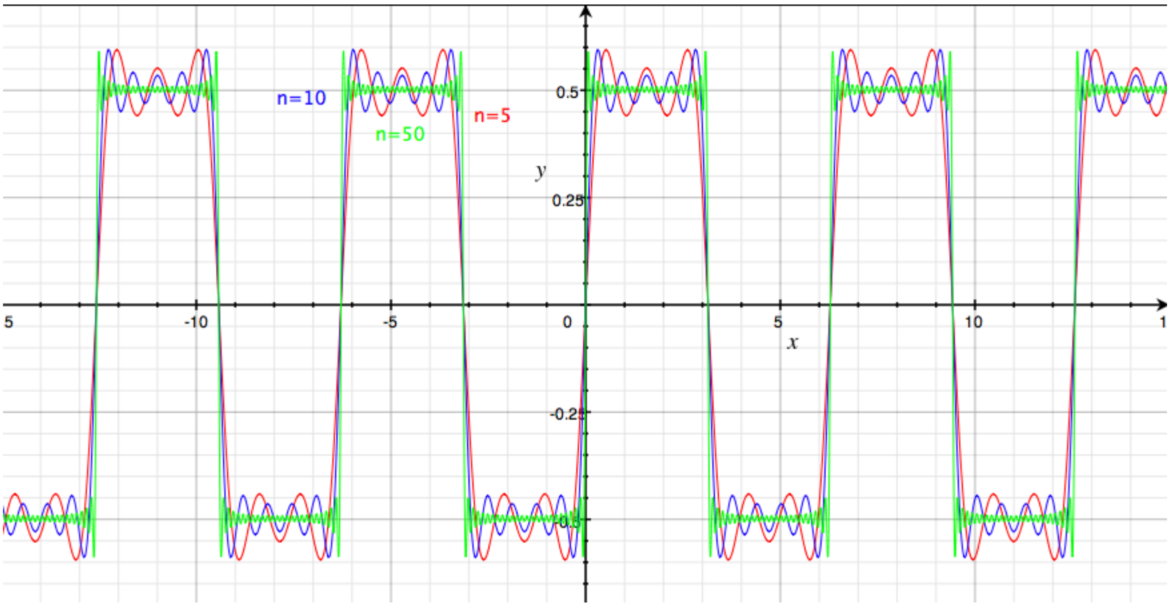

Figure 24: Fourier series for n=5,10,50

$$= \frac{1}{-2\pi i k}(1-\cos(k\pi)) = \begin{cases} \frac{1}{\pi k} & \text{if } k \text{ odd.} \\ 0 & \text{if } k \text{ even} \end{cases}$$

This implies if

$$f_n(x) = \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd} \\ k \leq n}} \frac{2}{\pi k} \sin(kx)$$

Some examples of this are shown in figure 24 Compute the nth partial Fourier series.

$$f_n(x) = \int_0^{2\pi} \underbrace{K_n(x-x')}_{\text{Dirichlet}} f(x')dx'$$

$$= \frac{1}{4\pi}\int_0^{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right)(x-x')\right)}{\sin((x-x')\pi)}dx' + \frac{1}{4\pi}\int_{\pi}^{2\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right)(x-x')\right)}{\sin((x-x')\pi)}dx'$$

If $\theta = M(x-x')$, $M = n + \frac{1}{2}$

$$\frac{1}{2\pi}\left(\int_{M(x-\pi)}^{Mx} \frac{\sin\theta}{2M\sin\left(\frac{\theta}{2M}\right)}d\theta - \int_{M(x+\pi)}^{-Mx} \frac{\sin\theta}{2M\sin\left(\frac{\theta}{2M}\right)}d\theta\right)$$

$$= \frac{1}{2\pi}\left(\int_{-Mx}^{Mx} \frac{\sin\theta}{2M\sin\left(\frac{\theta}{2M}\right)}d\theta - \int_{-M(x+\pi)}^{-M(\pi-x)} \frac{\sin\theta}{2M\sin\left(\frac{\theta}{2M}\right)}d\theta\right)$$

$$\underbrace{=}_{\text{integrand is even}} \frac{1}{2\pi}\int_{-Mx}^{Mx} \frac{\sin\theta}{2M\sin\left(\frac{\theta}{2M}\right)}d\theta - \int_{M\pi-Mx}^{M\pi+Mx} \frac{\sin\theta}{2M\sin\left(\frac{\theta}{2M}\right)}d\theta$$

We are interested in the behaviour of f_n close to the discontinuities (jumps) at $x \in k\pi$, $k \in \mathbb{Z}$ Both domains of integration have size $2Mx$ hence the size of the integral decides which integral dominates. If $|x| << 1$

4.10.1 First Integral

then $\theta = Mx$

$$\left|\sin\left(\frac{Mx}{2M}\right)\right| = \left|\sin\left(\frac{x}{2}\right)\right| \leq \frac{1}{2}|x| << 1$$

4.10.2 Second Integral

$\theta = M\pi + Mx$

$$\sin\big((M\pi + Mx)2M\big) = \sin\left(\frac{\pi}{2} + \frac{x}{2}\right) \to 1 \text{ as } x \to 0$$

This shows the second integral is bounded by:

$$\frac{2Mx}{2M\sin\left(\frac{\pi}{2}+\frac{x}{2}\right)} = \frac{x}{\sin\left(\frac{\pi}{2}+\frac{x}{2}\right)} \to 0 \text{ if } x \to 0$$

4.10.3 Maximal and Minimal values

For which values of x is the first integral maximal?

$$\frac{d}{dx}\int_{-Mx}^{Mx} \frac{\sin\theta}{2M\sin\left(\frac{\theta}{2\pi}\right)} d\theta$$

$$= \frac{\cancel{2M}}{\cancel{2M}} \frac{\sin(Mx)}{\sin\left(\frac{\cancel{M}x}{2\cancel{M}}\right)} = 0$$

If $x = \frac{k\pi}{M}$ for some $k \in \mathbb{Z}$

$$\lim_{M\to\infty} \frac{1}{4\pi M} \int_{-k\pi}^{k\pi} \frac{\sin\theta}{\sin\left(\frac{\theta}{2M}\right)} d\theta$$

$$\lim_{M\to\infty} \frac{1}{4\pi M} \int_{-k\pi}^{k\pi} \frac{\sin\theta}{\frac{\theta}{2M}} d\theta + \lim_{M\to\infty} \int_{-k\pi}^{k\pi} \sin\theta \left(\frac{1}{\sin\left(\frac{\theta}{2M}\right)} - \frac{1}{\frac{\theta}{2M}} \right) d\theta$$

$$= \lim_{M\to\infty} I_1 + \lim_{M\to\infty} I_2$$

we can expect $\lim_{M\to\infty} I_x = 0$ Consider first I_1.

$$\frac{1}{4\pi M} \int_{-k\pi}^{k\pi} \frac{\sin\theta}{\frac{\theta}{2M}} d\theta = \frac{1}{2\pi} \int_{-kx}^{kx} \frac{\sin\theta}{\theta} d\theta$$

As you can see in figure 25 Clearly I_1 is maximum for $k = 1$. Numerical integration of this yields:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin\theta}{\theta} d\theta \approx 0.59$$

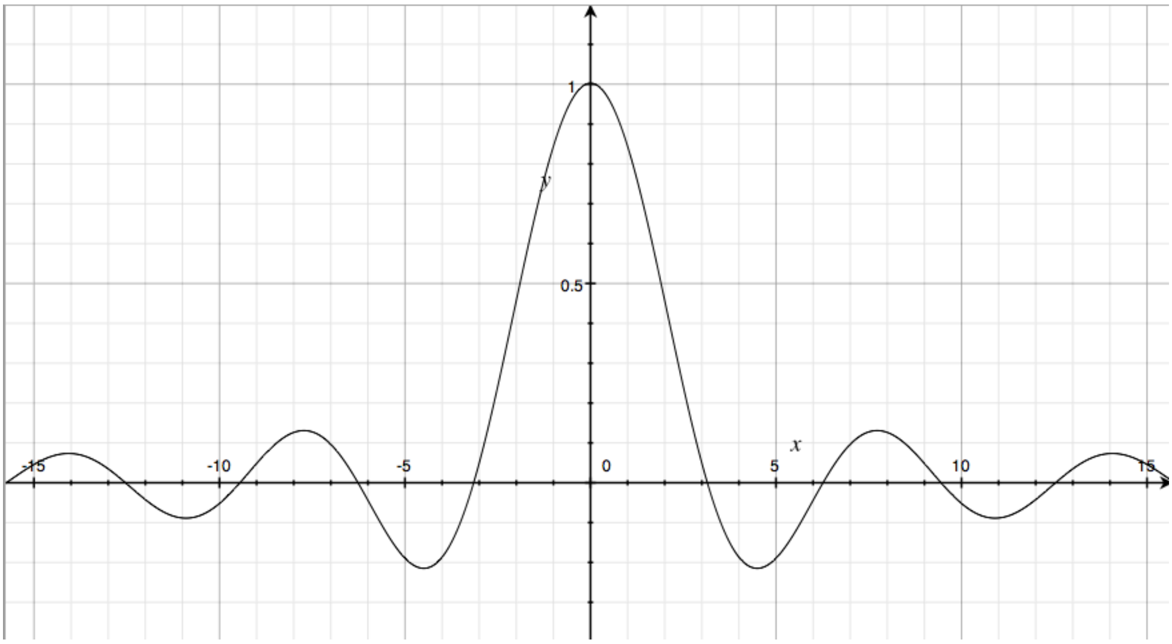

Figure 25: Graph of the function $\frac{\sin x}{x}$

Lecture 19:19/2/07

$$f_n(x) = \sum_{k=-n}^{n} e^{-ikx} \hat{f}(k)$$

$$\sup_{x\in\mathbb{R}} f_n(x) \geq 0.5y > 0.5 = \sup_{x\in\mathbb{R}} f(x)$$

This is correct we show that two integrals go to 0 as $n \to \infty$

$$J_1(M) = \int_{M(\pi-x)}^{M(\pi+x)} \frac{\sin\theta}{2M\sin\left(\frac{\theta}{2M}\right)} d\theta$$

$$J_2(M) = \frac{1}{4\pi M}\int_{-k\pi}^{k\pi} \sin\theta \frac{\frac{\theta}{2M} - \sin\left(\frac{\theta}{2M}\right)}{\sin\left(\frac{\theta}{2M}\right)\frac{\theta}{2M}} d\theta$$

Show

$$\lim_{M\to\infty} |I_1(M)| \underbrace{=}_{\substack{y=\theta-M\pi \\ x=\frac{k\pi}{M}}} \lim_{M\to 0}\left|\frac{1}{4\pi M}\int_{-k\pi}^{k\pi}\frac{\sin(M\pi - y)}{2M\sin\left(\frac{M}{2}+\frac{y}{2\pi}\right)}dy\right|$$

$$\leq \frac{k}{2M}\frac{1}{\sin\left(\frac{\pi}{2}+\frac{k}{2M}\right)} = 0$$

$$\lim_{M\to\infty} \frac{1}{4\pi M}|\sin\theta| \frac{\left|\frac{\theta}{2M} - \sin\left(\frac{\theta}{2M}\right)\right|}{\left|\sin\left(\frac{\theta}{2M}\right)\cdot\frac{\theta}{2M}\right|} \leq \frac{1}{4\pi M}\left|\frac{\theta}{2M} - \frac{\theta}{2M} + C(\theta)\frac{\theta^2}{4M^2}\right| \qquad (4.4)$$

As figure26 clearly shows

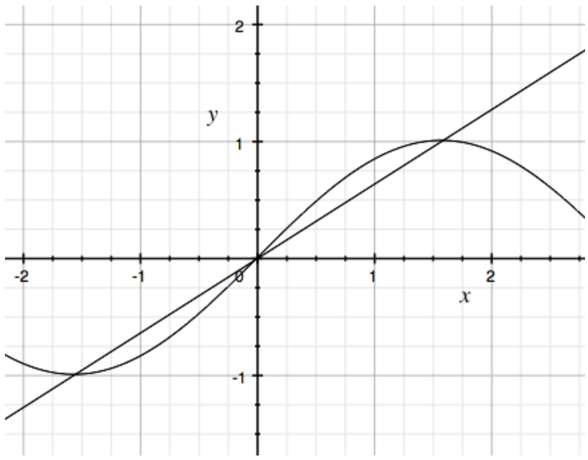

Figure 26: Graph of the function $\sin x$ and $\frac{2x}{\pi}$ showing sin is always bigger.

$$\left|\sin\left(\frac{\theta}{2\pi}\right)\right| \geq \left|\frac{1}{\pi}\frac{\theta}{2M}\right|$$

Therefore this means that equation 4.4 is:

$$\leq \frac{1}{4\pi M}\frac{\left|\frac{\theta}{2M} - \frac{\theta}{2M} + C(\theta)\frac{\theta^2}{4M^2}\right|}{\left|\frac{1}{\pi}\frac{\theta^2}{4M^2}\right|} \leq \frac{1}{4M}C(\theta)$$

Which tends to zero uniformly in θ as

$$C(\theta) = \cos\eta$$

for some number $\eta \in (0, \theta)$ it is therefore bounded uniformly in θ so the integral $\to 0$.

$J_2 \to 0$ uniformly in θ, since the length of domain of integration is finite and independent of M, this shows that

$$\lim_{M\to\infty} |I_2(M) = 0|$$

4.11 Back to PDE's

Recall that the motivation for the study of Fourier Series was the possibility to constant solutions of the wave equation using a separation Ansatz[1, 2]. We will first construct 2π-periodic solutions. This is often reffered to as "periodic boundary conditions." It is nice to use boundary conditions as you are in the periodic realm of Fourier series.

Separation Ansatz $u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}(k,t)e^{ikx}$

$$u_{tt} = c^2 u_{xx},\ u(x,0) = \sum_{k\in\mathbb{Z}} \hat{a}(k)e^{ikx},\ u_t(x,0) + \sum_{k\in\mathbb{Z}} \hat{b}(k)e^{ikx}$$

$\hat{a}(k), \hat{b}(k) \in \mathbb{C}$ are given. We want to find $\hat{u}(k,t)$/ Formally we obtain:

$$\sum_{k\in\mathbb{Z}} \ddot{\hat{u}}(k,t)e^{ikx} = \sum_{k\in\mathbb{Z}} k^2\hat{u}(k,t)e^{ikx}$$

Multiply with e^{ikx} and integrate from 0 to 2π and then we get:

$$\ddot{u}(k,t) = -k^3\hat{u}(k,t)$$

4.11.1 Putting it in terms of η and ξ.

$$\hat{u}(k,t) = \xi(k)cos(kt) + \eta(k)\sin(ckt)$$

Initial conditions:

$$\hat{u}(k,t) = \hat{a}(k)\cos(ckt) = \frac{\hat{b}(k)}{k}\sin(ckt)$$

If $\hat{b}(0) = 0$

$$u(x,t) = \sum_{k\in\mathbb{Z}} \big(\hat{a}(l)\cos(ckt) + \frac{\hat{b}(k)}{ck}\sin(ckt)\big)e^{ikx}$$

What happens for the heat equation if we apply this idea to the diffusion equation.[23]

$$u_t = \gamma u_{xx},\ u(x,0) = \sum_{k\in\mathbb{Z}} \hat{a}(k)e^{ikx}$$

$$\dot{\hat{u}}(k,t) = \gamma k^2\hat{u}(k,t) \Rightarrow u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}(k) = e^{\gamma ik^2t+ikx}$$

Lecture 20: 20/2/07

$$u(x,0) = \sum_{k\in\mathbb{Z}} \hat{a}_k e^{ikx}$$

$$u_t = \gamma u_{xx}$$

then $u_t(x,t) = \sum \hat{a}_k e^{-\gamma k^2 t}e^{ikx}$ In homogeneous problems:

$$u_t = \gamma u_{xx} + f(x,t)$$

Periodic boundary conditions:

$$\hat{u}_t(k,t) = -\gamma k^2\hat{u}(k,t) + f(k,t)$$

We can solve this ODE using Duhamels principle.

$$\hat{u}(k,t) = e^{-\gamma k^2 t}\hat{u}(k,0) + \int_0^t e^{-ck^2(t-s)}\hat{f}(k,s)ds$$

Now we need to check that:

$$\hat{u}_t(k,t) - \gamma k^2\hat{u}(l,0) + \hat{f}(k,t) - \gamma k^2\int_0^t e^{-\gamma k^2(t-s)}xf(k,s)ds$$

[24] And check that

$$\hat{u}_t(k,t) - \gamma k^2\hat{u}(l,0) + \hat{f}(k,t) - \gamma k^2\int_0^t e^{-\gamma k^2(t-s)}\hat{f}(k,s)ds$$

[23] γ replaces κ as it isn't a great choice of letter with all the x's and k's flying around.

[24] I think x should possibly be a gamma

Hence

$$u(x,t) = \sum_{k\in\mathbb{Z}} \left[e^{-\gamma k^2 t + ikx}\hat{u}(k,0) + \int_0^t e^{-\gamma k^2(t-s)+ikx}\hat{f}(k,s)ds \right]$$

Is the solution of the inhomogeneous diffusion equation.

4.11.2 Solutions of the homogeneous diffusion equation

Regularity of the solutions of the homogeneous diffusion equation. Lets assume that

$$u(x,0) \in C(\mathbb{R}) \text{ (cts)}$$

Then

$$\hat{u}(k,0) \leq D = \sup_{x\in\mathbb{R}} |u(x,0)|$$

A priori it is not clear whether $u(x,t) = \sum_{k\in\mathbb{Z}} e^{-\gamma k^2 t + ikx}\hat{u}(k,0)$ converges at all. We will show that $u \in C^\infty(\mathbb{R}\times(0,\infty))$. Let $s > 0$ then:

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} |\hat{u}(k,0)||k|^s e^{-\gamma k^2 t} \\
&\leq C \sum_{k=1}^{\infty} |k|^s e^{-\gamma k^2 t} \\
&\leq C \int_1^\infty e^{s\log k - \gamma(k-1)^2 t} \\
&\leq C \int_1^\infty e^{s(k-1)-\gamma(k-1)^2 t} dk \\
&\leq C \int_1^\infty e^{s(k-1)-\gamma(k-1)^2 t} dk \\
= C \int_0^\infty e^{sk - ck^2 t} dk &= D \int_0^\infty e^{-\gamma\left(k - \frac{s}{2\gamma}\right)^2 t \frac{1}{4\gamma} s^2 ds} \\
&\leq C e^{\frac{1}{4\gamma}^{25} s^2 t} \underbrace{\int_{-\infty}^\infty e^{-\gamma t k^2}}_{\leq \frac{1}{\sqrt{\gamma t}}\sqrt{\pi}} \leq \frac{D\sqrt{\pi}}{\sqrt{\gamma t}} e
\end{aligned}$$

We showed earlier [26] that $\sum_{k\in\mathbb{Z}} |\hat{f}(k)||k|^s < \infty$ implies that $\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx}$ converges to a limit f which is r times differentiable, (where $r < s-1$). This shows that the function

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{-\gamma k^2 t + ikx}\hat{u}(k,0)$$

Is infinitely often differentiable wrt, x if $t > 0$. Analogously one gets that $u \in C^\infty(\mathbb{R}\times(0,\infty))$. We convince ourselves that u satisfies the diffusion equation.

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial t}\sum_{k\in\mathbb{Z}} e^{-\gamma k^2 t + ikx}\hat{u}(k,0)$$

[26] Find the relevant equation.

And since eqch term is differentiable and the sum of the derivatives converges absolutely.

Analogously one obtains that:

$$\gamma\frac{\partial^2}{\partial x^2}u(x,t) = \gamma\frac{\partial^2}{\partial x^2}\sum_{k\in\mathbb{Z}} e^{-\gamma k^2 t+ikx}\hat{u}(k,0)$$

$$= \gamma \sim_{k\in\mathbb{Z}} -k^2 e^{-\gamma k^2 t+ikx}\hat{u}(k,0)$$

$$= \frac{\partial}{\partial x}u$$

.

4.11.3

We will show next that $u \in C(\mathbb{R}\times[0,\infty]$, if

$$|\hat{u}(k,0)| \leq \frac{\gamma}{|k|^4}$$

(this assumption is for the sake of simplicity, it isn't actually required though it makes calculations easier.)

$$|u(x,s)-u(y,t)| = \left|\sum_{k\in\mathbb{Z}}\hat{u}(k,0)\left(e^{-\gamma k^2 s_i k} - e^{-\gamma(t-iky)}\right)\right|$$

$$\leq \sum_{k\in\mathbb{Z}}|\hat{u}(k,0)|\left|e^{-\gamma k^2 s}e^{ikx} - e^{\gamma k^2 t}e^{ikx}\right| + \sum_{k\in\mathbb{Z}}|\hat{u}(k,0)|\left|e^{-\gamma i k^2 t}e^{ikx} - e^{\gamma k^2 t - iky}\right|$$

$$\leq \sum_{k\in\mathbb{Z}}\underbrace{|\hat{u}(k,0)|\gamma k^2|s-t|}_{D_1} + \sum_{k\in\mathbb{Z}}\underbrace{|\hat{u}(k,0)|k|x-y|}_{D_2} \leq D_1|s-t| + D_2|x-y|$$

Where $D_1, D_2 < \infty$ and $|\hat{u}(k,0)| < \frac{\gamma}{|k|^4}$

Lecture 21: 22/2/07

Last time on PDE's

$$u(x,t) = \sum \hat{u}(h,0)e^{-xk^2t+ixk}$$

is a 2π-periodic solution of: $u_t = \gamma u_{xx}$ and $u \in C^\infty(\mathbb{R}\times(0,\infty))$. The solutions of the wave equation behave differently

$$u(x,t) = \sum_{k\in\mathbb{Z}}\left(\hat{a}(k)\cos(\Upsilon kt)\frac{\hat{b}(k)}{k}\sin(\Upsilon kt)\right)e^{ikx}$$

In particular $u(x,0) = u(x,\frac{2\pi}{\Upsilon}$, hence the regularity of u does not improve as t increases. In contrast to the diffusion equation we can solve the equation backwards in time.

Homework Show that $u(x,t)$ solves the wave equation if:

$$|\hat{a}(k)| + |\hat{b}(k)| < \frac{\gamma}{|k|^4}$$

4.12 Other boundary conditions

Let $f \in C(0,\pi)$ Want

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(kx)$$

We extend f to an odd 2π periodic function. The fourier coefficients of g are:

$$\hat{g}(0) = \frac{1}{2\pi}\int_0^{2\pi} g(x)dx \underbrace{=}_{g \text{ is odd}} 0$$

$$\hat{g}(k) = \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-ikx}dx$$

Symmetry

$$\hat{g}(k) = \frac{1}{2\pi}\int^{2\pi} g(-x)e^{-ikx}dx$$

$$\underbrace{=}_{periodicity} -\frac{1}{2\pi}\int_0^{2\pi} g(2\pi - x)e^{ik(2\pi - x)}dx = \int_0^{2\pi} g(y)e^{iky}dy = -\hat{g}(-k)$$

$\Rightarrow \hat{g}(-k) = -\hat{g}(k)$

$$\sum_{k\in\mathbb{Z}} e^{ikx}\hat{g}(x) = \sum_{k=1}^{\infty} \hat{g}(k)\big(\cos kx - i\sin kx\big) - g'(k)\big(\cos kx - i\sin kx\big) = 2\sum_{k=1}^{\infty} i\hat{g}(k)\sin kx$$

Assume now that f is real valued this implies that $\hat{g}(-k) = -\hat{g}(k)$ this implies $\hat{g}(b) \in \mathbb{R}$ [27]

$$\Rightarrow \hat{g}(k) = \frac{i}{2\pi}\int^{2\pi} \sin kx g(x)dx$$

$$= \frac{-i}{\pi}\int_0^{\pi} \sin kx f(x)dx$$

$$\Rightarrow f(x) = \sum_{k=1}^{\infty} a_k \sin kx \quad x \in (0,\pi)$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx dx$$

This shows that $f(x)$ can be written as a sin series if $x \in (0,\pi)$. If $f(0) = f(\pi)$ then $g \in C(\mathbb{R})$, in this case:

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(kx) \; \forall x \in \mathbb{R}$$

[27] What b is I'm not sure.

4.13 Why do Fourier series work?

Recall from LA that if $A \in \mathbb{C}^{n\times n}$ Hermitian [28] $\Leftrightarrow$ symmetric if and only if $A \in \mathbb{R}^{n\times n}$ Let $(\cdot,\cdot)$ be an inner product since A is hermitian $(Ax, y) = (x, Ay)$ and a theorem from LA states that $\exists$ eigenvectors $v_1, \ldots, v_n \in \mathbb{C}^n$ (or $\mathbb{R}^n$) which form basis and are orthogonal $(\boldsymbol{v}_i, \boldsymbol{v}_j) = 0$ if $i \neq j$.

4.13.1 Infinite dimensional case

$(X, (\cdot,\cdot))$ inner product space $A : D \subset X \to$ X, where D is the domain of A, A is Hermitian (or symmetric) if $(Au, v) = (u, Av)$ $\forall u, v \in D$. We now choose:

$$X = C_{2\pi}(\mathbb{R}) \text{ (real or complex valued)}$$

and $D = C^2_{2\pi}(\mathbb{R})$[29] Inner product

$$(u, v) = \int_0^{2\pi} a(x)$$

(L^2-inner product) Operation on A, $Au = \Delta u = \frac{d^2}{dx^2}$ Now check A is hermitian

$$(Au, v) = \int_0^{2\pi} \overline{u''} v(x) dx = \int_0^{2\pi} \overline{u}'(x) v'(x) dx = \int_0^{2\pi} \overline{u(x)} v''(x) dx = (u, Av)$$

The equalities are due to integration by parts.

4.13.2 Eigenfunctions

An eigenfunction is a function f such that for a differential operator $\mathcal{A}$ and constant λ,

$$\mathcal{A} f = \lambda f \tag{4.5}$$

$$Au = \lambda u \Leftrightarrow u'' = \lambda u \Leftrightarrow u = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$$

u is not periodic if $\lambda > 0$ Hence

$$u(x) = C_1 e^{ikx} + C_2 e^{-ikx}$$

where $k = \sqrt{-\lambda} \geq 0$. If u is 2π-periodic, then $k \in \mathbb{Z}$. This shows that the trigonometric system. $(e^{ikx})_{k\in\mathbb{Z}}$ are the eigenfunctions of A, hence they form an orthogonal basis.

5 The Laplace Operator

Lecture 22: 5/3/07 [30] We finish this course by studying the most important partial differential equation:

[28] The lecturer describes this as a self-adjoint matrix, but fitting with Algebra 1 this means the matrix is Hermitian [3]

[29] $C_{2\pi}$ means it is 2π periodic.

[30] Thank's to Adam Kew for the lecture notes for this lecture because I overslept :o

Definition 5.1 (The Poisson equation)**.**

$$\nabla^2 u(x) = \Delta u(x) = f(x) \tag{5.1}$$

This equation has dirichlet boundary condition's:

$$f(x) = p(x),\ \ x \in \delta\Omega$$

where $\Omega \subset \mathbb{R}^n$ *is open,* $\delta\Omega = \bar{\Omega} \backslash \Omega$ [31]

$$f \in C(\Omega),\ \ g \in C(\delta\Omega), \Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i} u(\bar{x})$$

$$\begin{cases} u_{xxt}(x) & n = 1 \\ u_{xx}(x,y) + u_{yy}(x,y) & n = 2 \\ u_{xx}(x,y,z) + u + yy(x,y,z) + u_{zz}(x,y,z) & n = 3 \end{cases}$$

Definition 5.2 (The Laplace equation)**.** *If in equation 5.1* $p(x) = 0$ *then:*

$$\nabla^2 f(x) = \Delta f(x) = 0 \tag{5.2}$$

otherwise Poisson equation solutions of Laplace's equation are called "harmonic functions". The importance of equation 5.1 is a consequence from the fact that it describes a large variety of physical and mathematical phenomena.

Example 5.1. Electrostatics Maxwell's equations state that curl $\bar{E} = 0$ and div$E = -4\pi\rho$

$$\left(\ E_z^{(2)} - E_\rho^{(y)} = 0 \ \right)$$

where $\bar{E}(x,y,z)$ is the electric field and $\rho(x,y,z) \in \mathbb{R}$ is the charge density. curl$\bar{E} = 0$ implies that there exists an electric field potential $\phi(x,y,z)$ s.t. $E = \nabla\phi$. div$\bar{E} = -4\pi\rho$ implies that $\Delta\phi = -4\pi\rho$

Example 5.2. Analytic functions of a complex variable. Write $z = x + iy$ and $f(z) = u(z) + iv(z) = u(x+iy) + iv(x+iy)$ f is complex differential if and only if u, v satisfy the Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$ (CR) Solutions u and v of (CR) are necessarily harmonic, i.e.

$$u_{xx} + u_{yy} = 0$$

$$v_{xx} + v_{yy} = 0$$

Example 5.3. Imagine Brownian motion in a container $\Omega \subset \mathbb{R}^n$. This means that particles inside D move "completely randomly" until they hit the boundary when they stop. Divide $\delta\Omega$ into two pieces $\delta\Omega_1$ and $\delta\Omega_2$ Let $u(\bar{x})$ be the probability that a particle which begins at $\bar{x}$ stops at some point of $\delta\Omega_1$. It can be shown that $\Delta f(x) = 0$ in Ω and $f(x) = 1$ if $x \in \delta\Omega_1$, $f(x) = 0$ if $x \in \delta\Omega_2$

[31] Because due to my essay on the Boundary Element Method (which uses Laplace's equation) meaning I at least think I understand it, I have used my own notation for the next section.

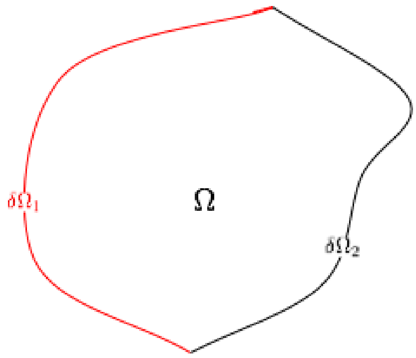

Figure 27: Diagram of a region Ω with the boundary split into two pieces.

Theorem 5.1 (Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $f \in C^2(\bar{\Omega})$ be harmonic in Ω. Then f achieves its maximum at some point $x \in \delta\Omega$. Then*

$$\max_{x\in\bar{\Omega}} f(x) = \max_{\tilde{x}\in\delta\Omega} f(\tilde{x})$$

Proof. Like in the case of the diffusion equation (equation 3.1) we prove the theorem by contradiction. Assume that there exists $x \in \Omega$ s.t. $f(x) > \max_{x'\in\delta\Omega} f(x')$ Since $f(x)$ is the maximum of f , the differentiability implies that $\nabla f = 0$ and the Hessian[32]

$$f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

i.e.

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \frac{\partial^2 f}{\partial x_i \partial x_j} \leq 0, \quad \forall \lambda \in \mathbb{R}^n \leq 0$$

Recall that:

$$\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x) = \text{Trace Hess}\big(f(x)\big) \tag{5.3}$$

33

$$= \sum_{i=1}^{n} \mu_i(x)$$

where μ_i are the eigenvalues of Hess$f(x)$ Suppose $\exists\, x \in \Omega$ s.t. $f(x) > f(\tilde{x})\ \forall \tilde{x} \in \delta\Omega$. This implies that $\nabla f(x) = 0$ and Hessian$(f) \leq 0$ in the sense of quadratic forms, this means that the eigenvalues are all non-positive. (Hess$(f)(x)$ are less than zero.)

[32] `http://en.wikipedia.org/wiki/Hessian_matrix`

[33] Trace is the sum of the diagonal entries, `http://en.wikipedia.org/wiki/Trace(linearalgebra)`

Lecture 23: 6/3/07 Recall from LA that since $\text{Hess}(f)(x) = \Delta f(x) = \sum_{i=1}^{n} \lambda_i \leq 0$. If we knew that $\Delta f(x) > 0$ then the proof would be finished. Since we haven't got this information we have to work a bit harder. Let

$$v(x) = u(x) + \epsilon|x|^2$$

then this implies that:

$$\Delta v(x) = \underbrace{\Delta u(x)}_{=0} + \epsilon \underbrace{\Delta|x|^2}_{\Delta(x_1^2+x_2^2+\cdots+x_n^2)} = 2n = 2\epsilon n > 0$$

The previous consideration implies now that v assumes it's maximum on $\delta\Omega$, i.e.

$$\max_{x\in\bar{\Omega}} = \max_{x\in\delta\Omega} v(x) \tag{5.4}$$

Now let $M_1 = \max_{x\in\delta\Omega} u(x)$, $M_2 = \max_{x\in\bar{\Omega}} u(x)$ By construction $M_1 \leq M_2$. So we want to show $M_1 = M_2$ hence it suffices to show that $M_1 \geq M_2$

$$M_2 \leq \big(\max x \in \bar{\Omega}\big)v(x) \underbrace{\leq}_{by\ equation\ 5.4} \big(\max x \in \delta\Omega\big)v(x)$$

$$\leq \max_{x\in\delta\Omega} u(x) + \epsilon R^2 = M + \epsilon R^2$$

where $R \in (0,\infty)$ has the property that $\Omega \in \{|x| < R\}$ $R < \infty$ since Ω is bounded. Since $\epsilon > 0$ is arbitrary this implies that $M_2 \leq M_1$. □

5.1 In variance of the Laplace operator

The laplace operator has the property that it doesn't change if we rotate the co-ordinate system. Let $n \in \{1,2,\ldots\}$ and $R \in \text{SO}(n)$[34] be a rotation matrix i.e.

$$R^T R = RR^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \det(R) = 1 = det(I)$$

$$\left\{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\right\} = SO(2)$$

It's easy to see that $R^{-1} = R^T$ is R is in $\text{SO}(n)$ (for some $n \in \mathbb{N}$)

5.1.1 Introduce the tilde'd variables

$$\tilde{x} = \underbrace{R}_{\text{rotation (in SO}(n))} x$$

We want to compute the representation of the Laplacian in the tilde'd variables.

First we compute the Hessian in the tilde'd variables.

$$\begin{aligned} \text{Hess}_x\Big(f\big(\tilde{x}(x)\big)\Big) &= \nabla_x \nabla_x^T\Big(f\big(\tilde{x}(x)\big)\Big) \\ &= \nabla_x\big(\nabla_{\tilde{x}} f(\tilde{x})R\big)^T \\ &= R^t \nabla_{\tilde{x}}^T \nabla_x f(\tilde{x}) R \\ &= R^T \text{Hess} S_x(f) R \end{aligned}$$

[34]SO means Special Orthogonal Group (or significant other, but not in this context ;))

Homework prove that $\text{Trace}\big(RAR^T\big) = \text{Trace}(A)$ if R is $SO(n),\ A = A^T$

$$\nabla_{\tilde{x}} f = \text{Trace Hess}_{\tilde{x}}(f)(\tilde{x})$$

$$= \text{Trace}(R)\text{Hess}_x(f)(\tilde{x})R^T$$

$$= \text{Trace Hess}_x(f) = \nabla_x(f) = \nabla_x f$$

This shows that the laplace operator is invariant and under rotations. This invariance suggests that the laplacian should take a particularly simple form in polar and spherical co-ordinates.

5.2 Laplacian in polar co-ordinates

$$x = r\cos\theta,\ y = r = \sin\theta$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$J^{-1} = \begin{pmatrix} \cos\theta & \frac{-\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix} = \begin{pmatrix} r_x & \theta_x \\ r_y\theta_y & \end{pmatrix}$$

where $\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\ \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$ Now square these operations.

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} \tag{5.5}$$

Lecture 24: 8/3/07

$$\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)^2 \tag{5.6}$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}r^2 + \frac{\cos\theta}{r}\frac{\partial}{\partial r}$$

Adding these operations we get:

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

It is natural to look for harmonic functions of the form $f(r,\theta) = f(r)$. These functions satisfy that $u_{rr} + \frac{1}{r}u_r = 0$ or equivalently

$$ru_{rr} + u_r = 0 \tag{5.7}$$

$= (ru_r)_r$

$$\Rightarrow ru_r = c_1 \Rightarrow u_r = \frac{c_1}{r}$$

$$\Rightarrow u(r) = c_1\log r + c_2$$

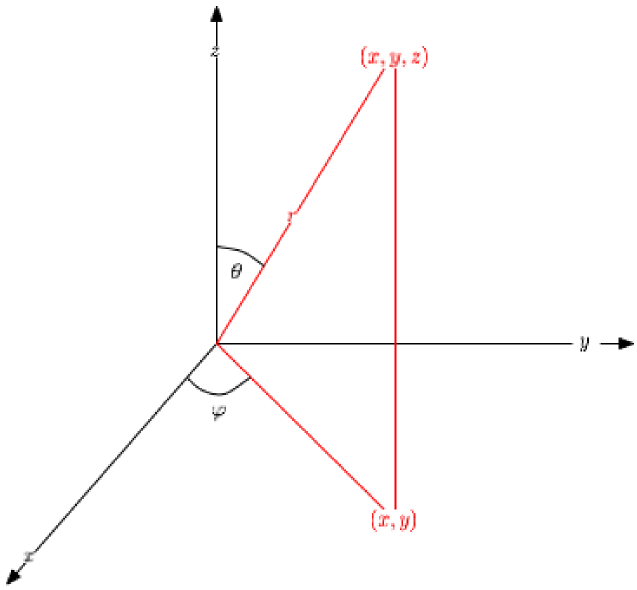

Figure 28: A reminder diagram of spherical coordinates.

In three dimensions we use spherical co-ordinates.

$$s = \sqrt{x^2 + y^2},\ r = \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2}$$

$$x = s\cos\varphi,\ y = s\sin\varphi,\ z = r\cos\theta, s = r\sin\theta$$

In spherical coordinates the Laplacian takes the form

$$\Delta_3 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

[35]

$$= f_{rr} + \frac{2}{r} f_r + \frac{1}{r^2 \sin\theta}(f_{\theta\theta} + \cos\theta f_\theta) + \frac{1}{r^2\sin^2\theta} f_{\varphi\varphi}$$

Let's look again for rotationally invariant solutions of the equations

$$\Delta_3 f = 0$$

$$f_{rr} + \frac{2}{r} f_r = 0 \Leftrightarrow (r^2 f_r)_r = 0$$

$$\Rightarrow f_r = \frac{c}{r^2} \Rightarrow f(r) = -\frac{c_1}{r} + c_2$$

This important harmonic function is the three dimensional analogue of the two dimensional function, $\log(x^2+y^2)$ found earlier. Strictly speaking neither function is finite at the

[35] In the lecture notes u is used instead of f and that is probably the more standard notation, oh well :o

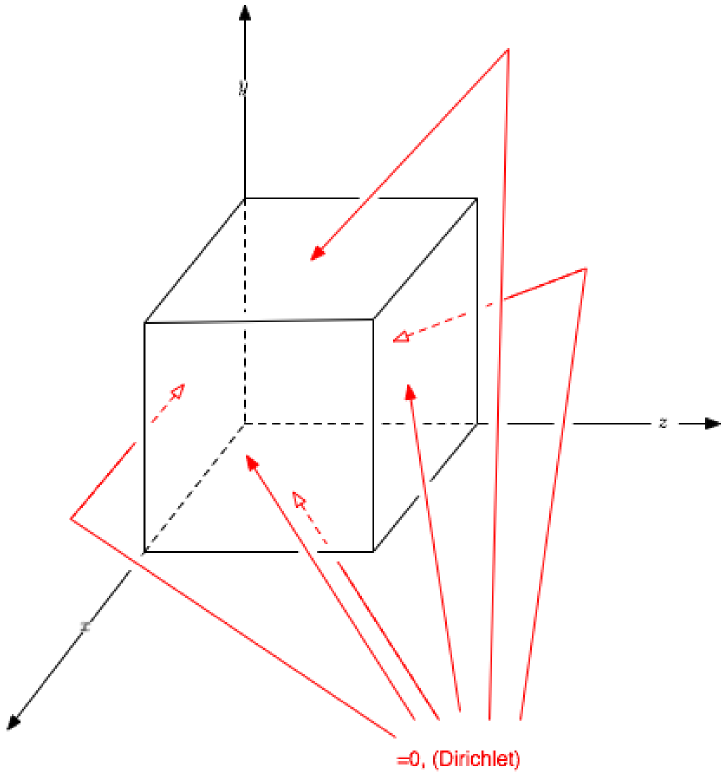

Figure 29: Diagram showing the boundary conditions.

original point. In electrostatics $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ turns out to be the electrostatic potential of a change sitting on the origin. Solutions of Laplace's equation via Fourier Series. If the domain D is simple enough and the boundary conditions are reasonable it is possible to solve Laplace's equation using separation of variables.

Example 5.4.

$$D = (0,\pi) \times (0,\pi) \times (0,\pi) \subset \mathbb{R}^3$$

In this case the boundary conditions[36] are:

$$f(0,y,z) = f(x,0,z) = f(x,\pi,z) = f(x,y,0) = f(x,y,\pi) = 0$$

$$f(\pi,y,z) = g(y,z)$$

This is shown in figure 29[37]. We now seek solutions of the form:

$$f(x,y,z) = a(x)b(y)c(z) \tag{5.8}$$

$$\Delta_3 f = 0 \Leftrightarrow a''bc + ab''c + abc'' = 0$$

$$\Leftrightarrow \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} = 0 \tag{5.9}$$

[36]these are Dirichlet boundary conditions

[37]It better be good, that damn diagram took AGES to draw.

The homogeneous BC's become $a(0) = b(0) = c(0) = a(\pi) = b(\pi) = c(\pi) = 0$ Each quotient has to be constant i.e

$$\frac{b''}{b} = \lambda,\ \frac{c''}{c} = \mu$$

Together with the boundary conditions we obtain that:

$$b(y) = B\sin(my),\ c(z) = C_1^r \sin(nz)$$

for $m, n \in \mathbb{N}$,$\lambda = m^2, \mu = n^2$. Now we have to deal with the equation for a:

$$a'' = \nu a,\ a(0) = 0$$

where

$$y = n^2 + m^2 > 0 \Rightarrow a(x) = A\sinh\left(x\sqrt{m^2+n^2}\right)$$

summing up the solution we get:

$$f(x, y, z) = \sum_{m,n=1}^{\infty} A_{m,n} \sinh\left(x\sqrt{m^2+n^2}\right)\sin(my)\sin(nz) \tag{5.10}$$

We have to find $A_{m,n}$ such that $f(\pi, y, z) = g(y, z)$ For each g we expand $g(y, z)$ into a sine series ie:

$$g(y, z) = \sum_{n=1}^{\infty} D_n(y)\sin(nz) \tag{5.11}$$

Repeat this step with

$$D_n g(y, z) = \sum_{m,n=1}^{\infty} D_{m,n}\sin(my)\sin(nz)$$

where

$$D_{m,n} = \frac{4}{\nu^2}\int_0^{\pi}\int_0^{\pi} g(y, z)\sin(ny)\sin(nz)dydz$$

and we obtain that

$$A_{mn} = \frac{4}{\pi^2\sinh\left(\sqrt{m^2+n^2}\pi\right)}\int_0^{\pi} dy\int_0^{\pi} dz \frac{dydz}{g(y, z)\sin(my)\sin(nz)} \tag{5.12}$$

A domain D is missed a bit and the boundary conditions [38]

Lecture 25:12/3/07 Recall from VA integral by parts:

$$\Omega \subset \mathbb{R}^n \quad \underline{v} : \Omega \to \mathbb{R}^n$$

vector field function is differentiable as necessary.

The product rule for vectors states:

$$\text{div}(f\underline{v}) = \nabla f \cdot \underline{v} + f \cdot \text{div}(v)$$

[38] I didn't quite get this bit of the notes

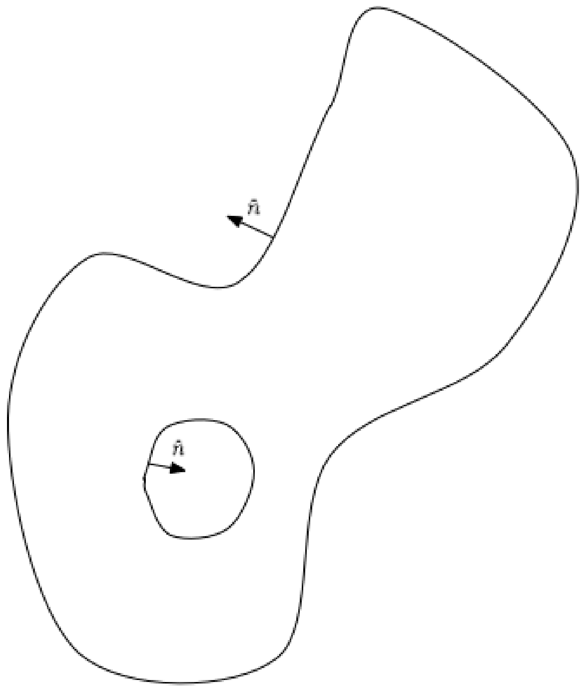

Figure 30: Diagram showing a region with a disconnected boundary

equivalently $\nabla \cdot (fv) = \nabla f \cdot \underline{v} + f\nabla\underline{v}$. Integrate and apply

$$\int_{\delta\Omega} f\underline{v} \cdot \underline{n} dS = \int_{\Omega} \nabla(f\underline{v}) dV = \int_{\Omega} \nabla f \cdot \underline{v} dV + \int_{\Omega} f\nabla\underline{v} dV$$

i.e.

$$\iiint_{\Omega} f\nabla\underline{v} \cdot \underline{n} dV = \iint_{\delta\Omega} f\underline{v} \cdot \underline{n} dS - \iiint_{\Omega} \nabla f \cdot \underline{v} dV \tag{5.13}$$

This is Green's first identity.

Definition 5.3 (Formal Definition of Green's 1st identity)**.** *Let $v = \nabla u$ $u : \Omega \to \mathbb{R}$ a function:*

$$\int_{\Omega} \nabla u \cdot \nabla f dV = \int_{\delta\Omega} f\underline{n} \cdot \nabla\underline{u} - \int_{\Omega} f\Delta u dV \tag{5.14}$$

$\delta\Omega$ doesn't have to be connected as can be seen in figure 30.

$$\Delta f = \nabla \cdot \nabla f$$

Sub mean value a property of subharmonic functions.

Definition 5.4. *$u : \Omega \to \mathbb{R}$ is subharmonic if $\Delta u(x) \geq 0$ $\forall x \in \Omega$. (Strictly convex functions, for example $|x|^2 = x_1^2 + \cdots + x_n^2$ have this property.*

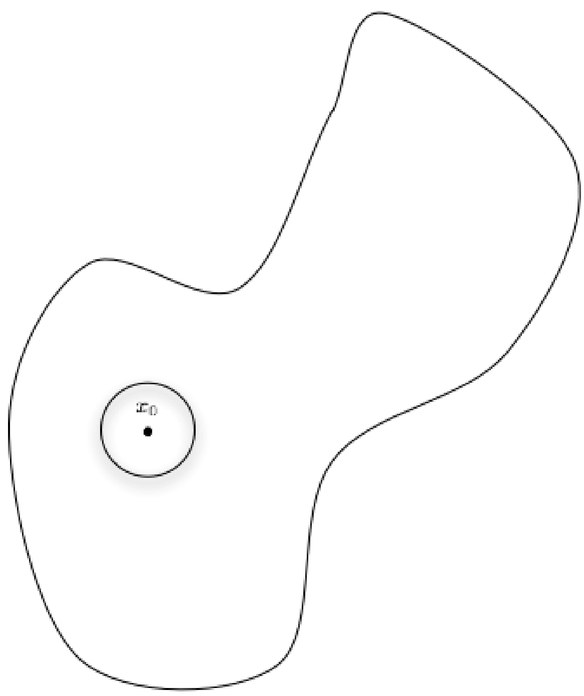

Figure 31: A ball inside a region Ω

Definition 5.5. *The (surface) mean value $\bar{u}_R$, of u on $B(x_0, R) \subset \Omega$ is defined by*

$$\bar{u}_R(x_0) = \frac{1}{4\pi R^2} \int_{\delta B(x_0,R)} u dS$$

($n = 3$) in this case.

Similar formulae exist for other values of n.

Definition 5.6. *$u : \Omega \to \mathbb{R}$ has the sub-mean value property if whenever $\overline{B(x_0, R)} \subset \Omega$ then $u(x_0) \leq \bar{u}_R$ i.e. u at the centre of a ball is less than or equal to the average value of u on the boundary of he ball. This is shown in figure 31.*

Theorem 5.2. *Hynothesis (price)*

1. *$u \in C^3(\Omega)$, $\Delta u \geq 0$*
2. *$B(x_0, R) \subset \Omega$*

Conclusion (reward) has the sub mean value property i.e $u(x_0) \leq \overline{u(x_0, R)}$

Proof. Apply equation 5.14 with $f \equiv 1$, $\Omega = B(x_0, R)$[39] then as $\nabla f = 0$.

$$\int_{\delta\Omega} \frac{\partial u}{\partial n} dS = \int_{B(x_0,R)} \Delta u \geq 0 \tag{5.15}$$

[39]Note that here are previously in this lecture Mario used the notation $B_R(x_0)$

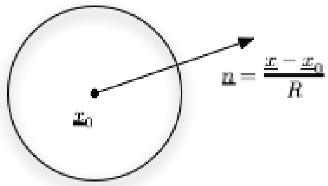

Figure 32: A ball of radius R with a normal vector $\underline{n}$

Then using polar coordinates centred at $\underline{x}_0$

$$\underline{x}_0 = (x_0, y_0, z_0)$$

$$\underline{x} = (x, y, z) = (x_0 + r\cos\phi\sin\theta, y_0 + r\sin\phi\sin\theta, z_0 + r\cos\theta)$$

for VA $dS = R^2 \sin\theta d\theta d\phi$

$$\frac{d}{dr} u(x_0 + r\cos\phi\sin\theta, y_0 + r\sin\phi\cos\theta, z_0 + r\cos\theta)\Big|_{r=R}$$

$$\frac{\partial u}{\partial x}(\underline{x})\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}(\underline{x})\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}(\underline{x})\frac{\partial z}{\partial r}$$

$\nabla u(\underline{x}) \cdot \underline{n} = \frac{\partial u}{\partial n}$ For $r > 0$

$$\frac{\partial}{\partial r}\left[\frac{1}{4\pi r^2}\int_{\delta B(x_0,r)} u dS\right]$$

[40]

$$\frac{\partial}{\partial r}\left(\frac{1}{4\pi \cancel{r^2}}\int_0^{2\pi}\int_0^{\pi} u(x_0 + r\cos\phi\sin\theta, y_0 + r\sin\phi\sin\theta, z_0 + r\cos\theta\right)\sin\theta \cancel{r^2} d\theta d\phi$$

$$\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}\frac{\partial u}{\partial n}(x_0) dS$$

which then by equation 5.15 is greater than zero. In particular $0 < \sigma < R$ then

$$\bar{u}_\sigma(x_0) \le \bar{u}_\sigma(x_0) \le \bar{u}_\sigma(x_0) \le \bar{u}_R(x_0) \tag{5.16}$$

As $\sigma \to 0$ $\bar{u}_\sigma(x_0) \to u(x_0)$, we claim $\lim_{\sigma\downarrow 0}\bar{u}_\sigma(x_0) = u(x_0)$ $\bar{u}_\alpha(x_0) \to u(x_0)$

Remark 5.1. *Claim* $\lim_{\sigma\downarrow 0}\bar{u}_\alpha(x_0) = u(x_0)$ *Assuming the claim the proof is complete so now prove the claim (easy by Analysis 2)*

Proof. Given $\epsilon > 0$ $\exists\delta > 0$ s.t. $|x - x_0| < 1 \Rightarrow |u(x) - u(x_0)| < \epsilon$ so if $0 < \alpha < \delta$

$$|u(x_0) - \bar{u}_\sigma(x_0)| = \left|\frac{1}{4\pi\sigma^2}\int_{\delta B(x_0,\sigma)} u(x_0) - u(\underline{x}) dS\right| \le \frac{1}{4\pi\sigma^2}\int_{\delta B(x_0,\sigma)} \epsilon dS = \frac{\cancel{4\pi\sigma^2}\epsilon}{\cancel{4\pi\sigma^2}} = \epsilon$$

□

[40] This is explained later basically it is $\frac{\partial}{\partial r}\bar{u}_R(x_0)$

So as the claim is correct the proof is now complete. □

Remark 5.2. *If u is harmonic the equality holds throughout the argument. $\geq 0 \Rightarrow 0$ everywhere mean value holds everywhere when $u(x_0) = \bar{u}_R(x_0)$ whenever $\overline{B(x_0, R)} \subset \Omega$*

Proof. u and $-u$ are both subharmonic if u is harmonic. □

Lecture 26:13/3/07 $d = 3$ $D \subset \mathbb{R}^3$ open $U \in C(D)$ subharmonic $\Rightarrow u(x_0) \leq \frac{1}{4\pi R^2} \int_{|x-x_0|} u(x)dS$ mean value inequality.

Remark 5.3. *The general mean value inequality*

$$u(x_0) \leq \frac{1}{\delta B(x, R)} \int_{\delta B(R)} u(\underline{x}')dx' \tag{5.17}$$

holds for all $u \in C(D)$ if $D \subset \mathbb{R}^d$ is open, $d \geq 1$, u is sub-harmonic. In particular $d = 2$ is possible.

Remark 5.4. *The sub mean value property means that u can bounded pointwise by integrals. For arbitrary functions u which satisfy differential (in)equality we only get inequalities where the supremum appears on the RHS e.g*

$$\int_{\delta B(x,R)} udS \leq \sup_{x \in \delta B} |u(x)||\delta B(x, R)|$$

Compare the mean value property of harmonic functions with Cauchy's integral formula

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\theta})d\theta = \frac{1}{2\pi i} \int_{|z-z_0|} \frac{f(z)}{z - z_0} dz$$

if $f : D \subseteq \mathbb{C}$ is holomorphic[41]. One application is:

Proposition 5.3 (Strong Maximum Principle)**.** *Let $D \subset \mathbb{R}^d$ be open, connected, bounded and $u \in C(D)$ be harmonic, then u achieves it's maximum in D if and only if u is constant.*

Proof. Let $x_m \in \bar{D}$[42] be the point where u is maximised. Assume that $x_m \in D$ then we have to show that u is constant. Find a ball $B \subset D$ centred at x_m. Since the average is not greater than the maxumum we have:

$$m := u(x_n) \underbrace{=}_{\text{mean value property}} \text{average on a sphere is less than or equal to m}$$

This implies now that $u(x) = m \ \ \forall x \in B$, every point in D can be reached with a finite number of spheres and this means $u(x) = m \ \forall x \in D$. Now we repeat this argument with a new centre on the sphere. We can fill up D with spheres since D is connected. This implies $u(x) = m$ thoughout, hence u is constant. □

[41] In three and higher dimensions there are no holomorphic functions as there are no complex numbers
[42] the closure of D

5.3 Dirichlet principle in connection with harmonic functions

This is a very very general property.

Theorem 5.4 (Dirichlet's principle)**.** *Let $D \subset \mathbb{R}^d$ be open and bounded and let $u \in C^1(\bar{D})$ be harmonic.*

$$\int_D |\nabla u(x)|^2 dx \leq \int_D |\nabla v(x)|^2 dx$$
$$\forall v \in C^1(\bar{D}) \text{ s.t. } v\big|_{\delta D} = u\big|_{\delta D}$$

($u(x) = v(x)\ \forall x \in \delta D$) The inequality is an equality if and only if: $u = v$.

Proof. Let $w = u - v$ then $w \in C^1(\bar{D})$ and $w\big|_{\delta D} = 0$

$$\Rightarrow \int_D |\nabla v|^2 dx = \int_D |\nabla v + \nabla w - \nabla w|^2 dx$$

$$= \int_D |\nabla u|^2 - 2\int_D (\nabla u, \nabla w) dx + \int_D |\nabla w|^2 dx$$

$$\underbrace{=}_{\text{Integration by parts}} \int_D |\nabla u|^2 dx + 2\int)D|w(x) \quad \underbrace{\nabla \cdot \big(\nabla u(x)\big)}_{\nabla(\nabla u)=\Delta u=0 \text{ as u is harmonic}} \quad dx$$

$$-\int_{\delta D} \underbrace{w(x)}_{=0} \nabla u(x) \cdot n(x) dS + \underbrace{\int_D |\nabla w|^2 dx}_{\geq 0} \geq \int_D |\nabla u|^2 dx$$

[43] □

Lecture 27:15/3/07

$$v\big|_{\delta D} = u\big|_{\delta D}, \quad \Delta u = 0$$

$$\int_D |\nabla v|^2 dx = \int_D |\nabla u|^2 dx + 2\int_D \omega \underbrace{\nabla \cdot \nabla u}_{=0} dx + \int \underbrace{|\nabla \omega|^2 dx}_{\geq 0} \geq \int_D |\nabla u|^2 dx$$

When is $\int_D |\nabla \omega|^2 dx = 0$? $v\omega(x) = 0\ \forall x \in D$, but then

$$\omega(x) = \underbrace{\omega(x')}_{=0} + \int_0^1 \frac{d}{dS}\omega\big(\gamma'(s)\big) dS$$

where $\gamma : [0,1] \to D$ s.t. $\gamma(0) = x' \in \delta D$ and $\gamma(1) = x$ we obtain that

$$\frac{d}{ds}\omega\big(\gamma'(1)\big) dS = \nabla\omega\big(\gamma(s)\big)\gamma'(s)$$

Types of Second Order differential equation. Let $d = 2$ and consider the general PDE

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = 0 \tag{5.18}$$

Most general second order differential equation is a scalar PDE with constant coefficients (as above).

[43] The Lecturer refers to the $\nabla\cdot$ as div but it's a bit inconsistent so I've used nabla

Theorem 5.5. *By a linear transformation of variables x and y the equation can be reduced to one of the following three forms.*

1. *Elliptic, if $a_{12}^2 < a_{11}a_{22}$ it is reducible to $u_{xx} + u_{yy} + \text{lot} = 0$. lot*[44] *contains the first and zeroth order derivatives.*
2. *Hyperbolic $a_{12}^2 > a_{11}a_{22}$ it can be reduced to $u_{xx} - u_{yy} + \text{lot} = 0$.*
3. *Parabolic, if $a_{12}^2 = a_{11}a_{22}$ it is reducable to $u_{xx} + \text{lot} = 0$*

Proof. Similar to analysis of conic sections. For simplicity let's assume $a_{11} = 1$ and $a_2 = a_1 = a_0 = 0$. By completing the squation we can write equation 5.18 as

$$(\partial_x + a_{12}\partial_y)^2 + (u_{22} - u_{12})\partial_y^2 = 0$$

In the elliptic case $a_{12} < a_{22}$, let $b = (a_{22}^2 - a_{11}^2)^{\frac{1}{2}} > 0$[45] Now we introduce new variables ξ and η by $x = \xi$ and $y = a_{12}\xi + b\eta$ then: $\partial_\xi = 1\partial_x + a_{12}\partial_y$, $\partial_\eta = 0\partial_x + b\partial_y$ so that the equation becomes $\partial_\xi^2 u + \partial_\eta^2 u = 0$ which is Laplace's equation. We used the notation $\partial_\eta = \frac{\partial}{\partial\eta}$ etc. etc. The other two cases are analogous. □

6 Notation

$u \in C^1(\bar{D})$ $D \subset \mathbb{R}^d$ is an open set δD is piecewise C^1 (eg Square), We say that $u \in C^1(\bar{D})$ satisfies Neumann bc's. If

$$\frac{\partial u}{\partial n}(x) = 0 \forall x \in \delta D$$

where $\frac{\partial u}{\partial n}(x) = \nabla u(x) \cdot \underline{n}(x)$

$$= \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(x)\underline{\underline{n(x)}} \text{ if } \underline{n}(x) \leq \mathbb{R}^d$$

is the outward pointing normal vector.[46]

[44]lot stands for lower order terms

[45]Lecturer had a_{22} without square

[46]There is also stuff about the exams, which we got in the final lecture, if you want it send me an email to matthew@matthewhutton.com

7 Acronyms/Abbeviations

BC - Boundary Conditions

CR - Cauchy Riemman

cts - continuous

FTC - Fundamental Theorem of Calculus

LA - Linear Algebra

lot - lower order terms

LHS - Left hand side.

ODE - Ordinary Differential Equation

PDE - Partial Differential Equation

RHS - Right hand side

RL - Riemann Lebersgue lemma

sa - self adjointation

SO - special orthogonal group.

VA - Vector Analysis

wlog - Without Loss of Generality

My email address is matthew.hutton.warwick@googlemail.com if you have any questions, corrections or comments.
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