MA3F1
Introduction to Topology

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Contents

1 Euler's Theorem

Definition 1. A *polyhedron* is a region in $\mathbb{R}^3$ bounded by a finite number of polygonal faces.

Definition 2. A *polyhedral surface* is a finite number of planar polygons in $\mathbb{R}^3$ s.t. each edge is the edge of exactly 2 faces and each vertex has a simple circuit of edges around it.

Definition 3.
- A *graph* is a connected subset of $\mathbb{R}^2$ made up of edges and vertices.
- A *tree* is a graph which contains no loops.

Lemma 1.1. *For a graph Γ, $V - E \leq 1$, with equality iff Γ is a tree.*

Proof. If Γ is a tree the $V - E = 1$ by induction on number of vertices:

$V = 1,\ E = 0$ then $V - E = 1$.
Suppose $V - E = 1$. Then adding a new edge must add a new vertex or a loop would be formed, and adding a vertex introduces one new edge.

If Γ is not a tree then it contains a loop. Removing an edge from the loop leaves Γ connected and increases $V - E$. Repetition of this will eventually produce a tree, so $V - E < 1$. □

Theorem 1.2 (Euler)**.** *A connected polyhedral surface K s.t. every simple circuit of straight pieces disconnects the surface has $V - E + F = 2$.*

Proof. Choose a maximal tree T in K (i.e. some of the edges and vertices of K s.t. adding any more would introduce a loop). Consider dual graph Γ to T. Γ has vertex in every face of K and edge crossing every edge of K which is not in T. Then Γ connected and does not disconnect K. If Γ is a tree then $V = V_T$, $E = E_T + E_\Gamma$, $F = V_\Gamma$, so

$$\begin{aligned} V - E + F &= (V_T - E_T) + (V_\Gamma - E_\Gamma) \\ &= 1 + 1 = 2 \quad \text{since } T, \Gamma \text{ are trees.} \end{aligned}$$

If Γ not a tree then contains circuit which not disconnect K, contrary to assumption. □

2 Topological spaces

Definition 4. $N \subset X$ is a neighbourhood of $x \in X$ if $\exists$ open U s.t. $x \in U \subset N$.

Lemma 2.1. *A set U is open iff it is/contains a neighbourhood of all of its points.*

Proof. $(\Rightarrow)$ U open so U is nhbd of each of its points.

($\Leftarrow$) Take $x \in U$ then $\exists\, N$, $x \in N \subset U$, N nhbd of x so exists open V_x, $x \in V_x \subset N \subset U$. Let $V = \bigcup_{x \in U} V_x$. This open and $V \subset U$. Clearly also $U \subset V$ by construction so $U = V$. □

Definition 5. $(X, \mathcal{T})$ topological space, $A \subset X$. $(A, \mathcal{U})$ is a topological space where

$$\mathcal{U} = \{U \cap A : U \in \mathcal{T}\}$$

called *induced topology* or *subspace topology.*

Definition 6. A point $x \in X$ is a *limit point* of $A \subset X$ if every nhbd of x meets A in at least 1 point not x.

Lemma 2.2. *A closed set C contains all of its limit points.*

Proof. Take $x \in X \setminus C$ then $x \in X \setminus C$ open, not meet C so x not a limit point. □

Definition 7. A any subset then $\overline{A} := A \cup \{\text{limit points of } A\}$.

Lemma 2.3. *$\overline{A}$ is closed and if C closed, $A \subset C$ then $\overline{A} \subset C$.*

Proof. Pick $x \in X \setminus \overline{A}$. x not a limit point of A so $\exists$ open U, $x \in U \subset X \setminus A$. Each point of U has an open set around it not meeting A (in fact U is such a set) so not a limit point of A, i.e. $U \subset X \setminus \overline{A}$. Let $C \supset A$, C closed. Then $X \setminus C$ open misses A so no point of $X \setminus C$ is a limit point of A. □

Definition 8. $A^\circ := X \setminus \overline{X \setminus A}$ (open). $\partial A := \overline{A} \setminus A^\circ$ (closed).

Definition 9. $D \subset \mathbb{R}^2$ is *starlike* if $\exists\, p \in \mathbb{R}^2$ s.t. each ray with endpoint p meets ∂D in a unique point and D is closed.

3 Compactness and Connectedness

Lemma 3.1. *Closed subset of a compact set is compact.*

Proof. $C \subset X$ closed, X compact. Let U_α open cover of C. Then $\{U_\alpha, X \setminus C\}$ open cover of X, so $\exists\, \alpha_1, \ldots, \alpha_t$ s.t. $\{U_{\alpha_1}, \ldots, U_{\alpha_t}, X \setminus C\}$ covers X, so $\{U_{\alpha_1}, \ldots, U_{\alpha_t}\}$ covers C. □

Lemma 3.2. *Continuous image of a compact space compact.*

IDEA: $\{U_\alpha\}$ open cover of Y, $\{f^{-1}(U_\alpha)\}$ open cover of X, find finite subcover, push forward to get finite subcover of Y.

Lemma 3.3. *A compact subset C of a Hausdorff space X is closed.*

Proof. Pick $x \in X \setminus C$. $\forall y \in C\ \exists$ open $U_y \ni x$, $V_y \ni y$, $U_y \cap V_y = \emptyset$. $\{V_y : y \in C\}$ open cover of C, compact, so $\{V_{y_1}, \ldots, V_{y_t}\}$ covers C. $U = \bigcap_{i=1}^t U_{y_i}$ open, $x \in U$, $U \cap \bigcup_{i=1}^t V_{y_i} = \emptyset$ so $U \cap C = \emptyset$, so $X \setminus C$ open. □

Lemma 3.4. *A continuous bijection f from compact X to Hausdorff Y is a homeomorphism.*

Proof. RTP f^{-1} cts, i.e. U open $\Rightarrow$ $f(U)$ open. Equivalently f closed: Let $C \subset X$ closed, so compact, $f(C)$ compact, so closed. □

Proposition 3.5 (Bolzano-Weierstrass)**.** *An infinite subset A of compact X has a limit point.*

Proof. Suppose not. Then $\forall\, x \in X$, x not a limit point of A so $\exists$ open $U_x \ni x$ s.t. $A \cap U_x = \{x\}$ or $\emptyset$. Then $\{U_x : x \in X\}$ open cover of X, no finite subcover as no finite number can cover A. Contradicts compactness. □

Lemma 3.6 (Lebesgue)**.** *Let U_α be open cover of compact metric X. Then $\exists\, \delta > 0$ s.t. each $B_\delta(x)$, $x \in X$, is contained in one of the U_α. δ is called the* mesh *of the cover.*

Proof. Suppose not. Then $\exists A_1, A_2, \ldots$ subsets of X, diameters tending to zero, none contained in any U_α. $\forall\, n$ pick $x_n \in A_n$. Either $\{x_n\}_{n=1}^{\infty}$ has only finitely many distinct points or is infinite, so has limit point by Bolzano-Weierstrass. Denote infinitely repeated point or limit point by p.

Let U_{α_i} s.t. $p \in U_{\alpha_i}$. Let $\varepsilon > 0$ s.t. $B_\varepsilon(p) \subset U$ and pick N s.t.

- $\mathrm{diam}(A_N) < \frac{\varepsilon}{2}$ and
- $x_N \in B_{\frac{\varepsilon}{2}}(p)$.

Then $d(x_N, p) < \frac{\varepsilon}{2}$ and $d(x, x_N) < \frac{\varepsilon}{2}\ \forall\, x \in A_N$. Then $d(x,p) < \varepsilon$ if $x \in A_N$ so $A_N \subset U_{\alpha_i}$, contradicting choice of $\{A_n\}$. □

Definition 10. $\mathcal{B} \subset \mathcal{T}$ is called a *basis*for $\mathcal{T}$ if every set in $\mathcal{T}$ is a union of sets in $\mathcal{B}$.

Definition 11. Given top. spaces $(X, \mathcal{U}), (Y, \mathcal{T})$ then $X \times Y$ is a topological space with topology defined by the basis

$$\mathcal{B} = \{U \times V : U \in \mathcal{U}, V \in \mathcal{T}\}.$$

Theorem 3.7. *X, Y compact $\Rightarrow$ $X \times Y$ compact.*

Proof. Can assume open cover is of form $U \times V$ where $U \subset X$, $V \subset Y$ open. Define

$$\begin{aligned} i_x &: Y \to X \times Y \quad i_x(y) = (x,y) \\ j_j &: X \to X \times Y \quad j_y(x) = (x,y) \end{aligned}$$

i_x continuous, so for $x \in X$, $i_x(Y) \subset X \times Y$ is compact, so $\exists$ finite subcover $\mathcal{F}_x = \{U_1 \times V_1, \ldots, U_t \times V_t\}$ which covers $i_x(Y)$. W.l.o.g. assume $x \in U_i$ for all i.

Define $U_x = U_1 \cap \cdots \cap U_t$. $\{U_x : x \in X\}$ is open cover of X so is finite subcover $\{U_{x_1}, \ldots, U_{x_s}\}$ of X. Then $\bigcup_{i=1}^{s} \mathcal{F}_{x_i}$ covers $X \times Y$. □

Definition 12. X is *disconnected* if exists surjective map $f\colon X \to \{0,1\}$.

Proposition 3.8. *X disconnected iff X contains non-trivial clopen subset.*

Proof. ($\Rightarrow$) $\{0\},\{1\}$ both open in $\{0,1\}$ so $f^{-1}(0), f^{-1}(1)$ open subsets of X. $f^{-1}(0) = X \setminus f^{-1}(1)$ so both closed.

($\Leftarrow$) $A \subset X$ clopen, $A \neq \emptyset, X$. Then define $f\colon X \to \{0,1\}$ by $f(A) = 0$, $f(X \setminus A) = 1$. ☐

Example 1. $\mathbb{Q} \subset \mathbb{R}$ disconnected. Required map is

$$x \mapsto \begin{cases} 0 & x < \pi \\ 1 & x > \pi. \end{cases}$$

Lemma 3.9. *Continuous image of connected space is connected.*

Proof. Let $f\colon X \to Y$ surj, cts, X connected. Suppose Y disconnected. Then $\exists\, g\colon Y \to \{0,1\}$ surjective map. Then $g \circ f\colon X \to \{0,1\}$ surjective map. Contradiction. ☐

Lemma 3.10. *A path connected space is connected.*

Proof. Suppose X pc, not connected, so $\exists\, f\colon X \to \{0,1\}$ surj map. Choose $x, y \in X$ s.t. $f(x) = 0, f(y) = 1$. Let $g\colon I \to X$ be a path from x to y. Then $f \circ g\colon I \to \{0,1\}$ surj map. Contradiction as I connected. ☐

Definition 13 (Path Algebra)**.**

- (Addition) $f, g\colon I \to X$. f path x to y, g path y to z. Then $f \cdot g\colon I \to X$ path x to z, $t \mapsto \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$
- (Inversion) $f\colon I \to X$ path from x to y. Then $\overline{f}\colon I \to X$ path y to x, $t \mapsto f(1-t)$.
- (Constant path) For $x \in X$, define $c_x\colon I \to X$ by $t \mapsto x$ is *constant path* at x.

Lemma 3.11. *$x, y \in X$. Say $x \sim y$ if $\exists$ path x to y. Then $\sim$ is an equivalence relation.*

Definition 14. The equivalence classes of this relation are called *path components* of X.

Lemma 3.12. *Open connected subset of $\mathbb{R}^n$ is path connected.*

Proof. $X \subset \mathbb{R}^n$ open, connected. Let $x \in X$ then $\exists\, \varepsilon$ s.t. $B_\varepsilon(x) \subset X$. This convex so path connected, so if Q is path component of x then $B_\varepsilon(x) \subset Q$. Hence Q nhbd of x so open in $\mathbb{R}^n$, $X \setminus Q$ union of other path components so also open. Hence Q is clopen so by connectedness $Q = X$. ☐

Lemma 3.13. *Continuous image of path connected space is path connected.*

Proof. $f\colon X \to Y$ surj map, X pc. Pick $a, b \in Y$, $c, d \in X$ s.t. $f(c) = a, f(d) = b$. Then X pc gives $g\colon I \to X$ path c to d. Then $f \circ g\colon I \to Y$ path a to b in Y. □

Corollary 3.14. *X, Y pc $\Rightarrow X \times Y$ pc.*

Proof. Let $(a, b), (c, d) \in X \times Y$. i_b, j_c continuous, images contain (a, b) and (c, d) resp., meet at (c, b). □

Proposition 3.15. *X connected $\Rightarrow \overline{X}$ connected.*

Proof. Let U, V open, disjoint s.t. $\overline{X} \subset U \cup V$. Hence $X = (X \cap U) \cup (X \cap V)$. One of $(X \cap U), (X \cap V)$ empty as X connected. W.l.o.g. $X \cap U = \emptyset$. Then $U \subset \overline{X} \setminus X$ is open, hence empty. Therefore $\overline{X}$ connected. □

Proposition 3.16. *Topologist's sine curve is connected by not path connected.*

Proof. Let $X = A \cup B$ where $A = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : 0 < x \leq 1\right\}$, $B = \{(0, y) : -1 \leq y \leq 1\}$.

- X connected: A, B cts images of intervals so pc. Suppose $f\colon X \to \{0, 1\}$ surj map. W.l.o.g. let $f(A) = 0$. Then $f(B) = 1$ but $\{1\} \subset \{0, 1\}$ open so $f^{-1}(1) \subset \mathbb{R}^2$ open, contains B and therefore points of A. Contradiction.
- X not pc: Let $f\colon I \to X$ be path $(0, 0)$ to $(1, \sin 1)$. Let $t = \sup\{f^{-1}(B)\}$. By continuity $f(t) \in B$. Examining a standard nhbd N of $f(t)$ in X shows that $N \cap X$ has infinitely many path components. By continuity f maps $[t, t + \delta)$ into N with $f(t) \in B$, $f(t + \delta) \in A$ so have path in $N \cap X$ from B to A which are in different path components. Contradiction. □

4 Identification Spaces

Definition 15. Let X be top space, $\sim$ equiv relation on X. Let $\frac{X}{\sim}$ set of equivalence classes, projection $p\colon X \to \frac{X}{\sim}$, $x \to [x]$.

Topologise $\frac{X}{\sim}$ by $U \subset \frac{X}{\sim}$ open $\iff p^{-1}(U) \subset X$ open. This is *identification topology*. Is maximal topology s.t. p cts.

Definition 16. A surjective map $f\colon X \to Y$ is called *identification map* if $U \subset Y$ open $\iff f^{-1}(U) \subset X$ open.

Proposition 4.1. *Let $f\colon X \to Y$ be identification map. Define $\sim$ on X by $x \sim y \iff f(x) = f(y)$. Then $\exists$ homeomorphism*

$$q\colon \frac{X}{\sim} \to Y, \quad q([x]) = f(x).$$

Proof. $q \circ p = f$.

$$\begin{aligned} U \subset Y \text{ open} &\iff f^{-1}(U) \subset X \text{ open} \\ &\iff p^{-1}(q^{-1}(U)) \subset X \text{ open} \\ &\iff q^{-1}(U) \subset \frac{X}{\sim} \text{ open.} \end{aligned}$$ □

Lemma 4.2. *An open, surjective map is an identification map.*

Proof. $f\colon X \to Y$ surj, open.

$$\begin{aligned} U \subset Y \text{ open} &\Rightarrow f^{-1}(U) \text{ open} \quad (f \text{ cts}) \\ &\Rightarrow f(f^{-1}(U)) = U \text{ open} \quad (f \text{ open}). \end{aligned}$$

□

Lemma 4.3. *A closed, surjective map is an identification map.*

Proof.

$$\begin{aligned} U \subset Y \text{ open } &\Rightarrow f^{-1}(u) \text{ open as } f \text{ cts} \\ &\Rightarrow f^{-1}(Y \setminus U) = X \setminus f^{-1}(U) \text{ closed} \\ &\Rightarrow f(X \setminus f^{-1}(U)) = Y \setminus U \text{ closed} \\ &\Rightarrow U \text{ open.} \end{aligned}$$

□

Lemma 4.4. *A surjective map f from a compact space X to a Hausdorff space Y is an identification map.*

Proof.

$$\begin{aligned} C \subset X \text{ closed} &\Rightarrow C \text{ compact} \\ &\Rightarrow f(C) \text{ compact (cts image of compact space)} \\ &\Rightarrow f(C) \text{ closed (compact subset of Hausdorff space)} \\ &\Rightarrow f \text{ closed.} \end{aligned}$$

□

Definition 17. X, Y spaces, $A \subset X$, $f\colon A \to Y$ a map. Define

$$X \cup_f Y = \frac{X \amalg Y}{\sim}$$

where $x \sim y$ if

- $x = y$ or
- $x, y \in A$, $f(x) = f(y)$ or
- $x \in A$, $y = f(x)$ or
- $y \in A$, $x = f(y)$.

Lemma 4.5. *$f\colon X \to Y$ map, $\sim$ equiv relation on X. If have $q\colon \frac{X}{\sim} \to Y$ s.t. $q \circ p = f$ where $p\colon X \to \frac{X}{\sim}$ natural projection then q cts.*

Proof. $U \subset Y$ open. $f^{-1}(U) \subset X$ open as f cts, i.e. $p^{-1}(q^{-1}(U)) \subset X$ open so $q^{-1}(U) \subset \frac{X}{\sim}$ open by defn of quotient topology. □

5 Fundamental Group

Definition 18. For a given basepoint $\star \in X$, a *loop* is a path in X from $\star$ to $\star$.

Definition 19. Two maps $f, g\colon X \to Y$ are *homotopic* if $\exists$ map $F\colon X \times I \to Y$ s.t.

$$F(x,0) = f(x) \quad F(x,q) = g(x)$$

write $f \simeq_F g$.

Lemma 5.1. *Homotopy is an equivalence relation on set of maps $X \to Y$.*

Definition 20.

- A *pair* (X, A) is a space X and a subspace $A \subset X$.
- A *map of pairs* $f\colon (X,A) \to (Y,B)$ is a map $f\colon X \to Y$ s.t. $f(A) \subset B$.

  Example 2. A loop is a map $(I, \{0,1\}) \to (X, \{\star\})$.

- $f, g\colon (X,A) \to (Y,B)$ are *homotopic* if $f \simeq_F g$ where $F(A \times I) \subset B$.
- $[X; Y]$ is set of homotopy classes of maps $X \to Y$.

Lemma 5.2. *$Y \subset \mathbb{R}^n$ convex. Then $|[X;Y]| = 1$ (any two maps $X \to Y$ are homotopic).*

Proof. $f, g\colon X \to Y$, then $f \simeq_F g$ where

$$F(x,t) = (1-t)f(x) + tg(x).$$

□

Theorem 5.3. *Let X top space, $\star$ a basepoint. Then*

$$\pi_1(X,\star) = [(I, \{0,1\}); (X, \{\star\})]$$

forms a group called the fundamental group *of X based at $\star$.*

Proof. IDEA: Define piecewise linear maps $q\colon I \to I$. I convex so these homotopic to identity or constant map as required by lemma.

- (Closure) Given $\alpha, \beta \in \pi_1(X,\star)$ represent α by $f\colon I \to X$, β by $g\colon I \to X$. Define $\alpha\beta = [f \cdot g]$ (path addition). Well defined: if $f \simeq_{f_t} f'$, $g \simeq_{g_t} g'$ then $f \cdot g \simeq_{f_t \cdot g_t} f' \cdot g'$.
- (Associativity) Define $q\colon I \to I$ by $q(0) = 0, q\left(\frac{1}{4}\right) = \frac{1}{2}, q\left(\frac{1}{2}\right) = \frac{3}{4}, q(1) = 1$ linear elsewhere. Then

  $$(f \cdot g) \cdot h = f \cdot (g \cdot h) \circ q$$

  but I convex so $q \simeq Id_I$ by straight line homotopy.
- (Unit) Define $e \in \pi_1$ by $[c_\star]$. Define $q\colon I \to I$ by $q(0) = 0, q\left(\frac{1}{2}\right) = q(1) = 1$, linear elsewhere. Then $f \circ q = f \cdot c_\star$. I convex...
- (Inverses) $\alpha = [f]$, define $\alpha^{-1} = [\overline{f}]$. Define $q\colon I \to I$ by $0, 1 \mapsto 0$, $\frac{1}{2} \mapsto 1$. Then $q \simeq_{q_t} c_0$ by straight line homotopy, so

  $$f \cdot \overline{f} = f \circ q \simeq_{f \circ q_t} f \circ c_0 = c_\star.$$

□

5.1 Properties of fundamental group

Map of based spaces induces homomorphism $f_*\colon \pi_1(X,\star_X) \to \pi_1(Y,\star_Y)$ defined by $f_*[\alpha] = [f \circ \alpha]$.

Proposition 5.4. *f_* has functorial properties (distributes over composition, identity induces identity).*

Proposition 5.5. *If $f\colon X \to Y$ homeomorphism then f_* is isomorphism.*

Proof. Let $g = f^{-1}$. Then

$$g_* \circ f_* = (g \circ f)_* = (Id_X)_* = Id_{\pi_1(X)}$$

where outside two equalities are due to functorial properties. Similarly $f_* \circ g_* = Id_{\pi_1(Y)}$. □

Proposition 5.6. *If $f \simeq_{f_t} f'$ then $f_* = f'_*$.*

Proof.

$$\begin{aligned} f_*[\alpha] &= [f \circ \alpha] \\ &= [f' \circ \alpha] \quad (f \circ \alpha \simeq_{f_t \circ \alpha} f' \circ \alpha) \\ &= f'_*[\alpha]. \end{aligned}$$

□

Proposition 5.7. *Let $X \subset \mathbb{R}^n$ be starlike from $\star$. Then $\pi_1(X,\star) = 0$.*

Proof. Any loop homotopic to constant loop by straight line homotopy. □

Definition 21. A path connected space with trivial fundamental group is called *simply connected*.

Lemma 5.8 (Path-Lifting)**.** *Given a path $f\colon I \to S^1$, $q \in \mathbb{R}$ s.t. $e(q) = f(0)$, where $e\colon \mathbb{R} \to S^1$ is the map defined by $x \mapsto e^{2\pi ix}$ then there exists a unique* lift *$\tilde{f}\colon I \to \mathbb{R}$ s.t. $\tilde{f}(0) = q$ and $f \circ \tilde{f} = f$.*

Definition 22. Suppose $f\colon I \to S^1$ is a loop at 1, $\tilde{f}$ lift of f with $\tilde{f}(0) = 0$. Then

$$w := \tilde{f}(1) \in \mathbb{Z}$$

is called *winding number* of f.

Lemma 5.9 (Homotopy-Lifting)**.** *Suppose $f\colon I^2 \to S^1$ map and $q \in \mathbb{R}$ has $e(q) = f(0,0)$. Then there exists a unique* lift *$\tilde{f}\colon I^2 \to \mathbb{R}$ s.t. $\tilde{f}(0,0) = q$ and $e \circ \tilde{f} = f$.*

Theorem 5.10. $\pi_1(S^1, \{1\}) \cong (\mathbb{Z}, +)$.

Proof. By path-lifting lemma (5.8) can define winding number as a map from {loops based at 1} to $\mathbb{Z}$. Can use homotopy-lifting lemma (5.9) to show it is well defined on homotopy classes. It then suffices to show it is bijective and a homomorphism.

- (Homomorphism) Let $f, g\colon I \to S^1$ be loops in with lifts $\tilde{f}, \tilde{g}$ starting at 0. Let $\tilde{g}' = \tilde{g} + w(f)$ so $\tilde{g}'$ starts where $\tilde{f}$ finishes. Then $\widetilde{f \cdot g} = \tilde{f} \cdot \tilde{g}'$ and

$$w(f \cdot g) = \widetilde{f \cdot g}(1) = \tilde{g}'(1) = \tilde{g}(1) + w(f) = w(g) + w(f).$$

- (Surjective) Define $f_n\colon I \to S^1$, $t \mapsto e^{2\pi i t n}$, $n \in \mathbb{Z}$. Then $\tilde{f}_n(t) = nt$ and $w(f_n) = n$.
- (Injective) Suppose $w(f) = 0$. Then $\tilde{f}$ a loop at 0, so homotopic to c_0 by straight line homotopy, so $f = e \circ \tilde{f} \simeq e \circ c_0 = c_1$. □

5.2 Calculation of $\pi_1(S^2)$

Lemma 5.11. *Suppose $X = U \cup V$, U, V simply connected, $U \cap V$ path connected, $\star \in U \cap V$, then $\pi_1(X, \star) = 0$.*

Proof. $f\colon I \to X$ loop. By Lebesgue's lemma (3.6) can cut I into pieces $I_1 \cup \cdots \cup I_t$ s.t. each piece mapped into either U or V or both. W.l.o.g. can assume $p_i = I_i \cap I_{i+1}$ (one point) mapped into $U \cap V$ (if not it contains either I_i or I_{i+1} and would not have needed to make the divisions so fine). Let $f_i = f|_{I_i}$. Then $f = f_1 \cdot f_2 \cdot \cdots \cdot f_t$. Choose path g_i in $U \cap V$ from p_i to $\star$. Then

$$f = f_1 \cdot f_2 \cdot \ldots \cdot f_t \simeq (f_1 \cdot g_1) \cdot (\overline{g}_1 \cdot f_2 \cdot g_2) \cdot \ldots \cdot (\overline{g}_{t-1} \cdot f_t)$$

where each bracket is a loop in either U or V or both, so is $\simeq c_\star$, so $f \simeq c_\star$. □

Proposition 5.12.

$$\pi_1(S^2) = 0.$$

Proof. Let

$$U = \left\{ \mathbf{x} \in S^2 : x_3 < \frac{1}{2} \right\}$$
$$V = \left\{ \mathbf{x} \in S^2 : x_3 > \frac{-1}{2} \right\}$$

Stereographic projection from $(0, 0, 1)$ maps U homeomorphically to convex subset of $\mathbb{R}^2$ so U simply connected, similarly V. $U \cap V$ pc, pick $\star \in U \cap V$, e.g. $(1, 0, 0)$ result by lemma. □

Lemma 5.13. *X pc, $x, y \in X$. Then $\pi_1(X, x) \cong \pi_1(X, y)$.*

Proof. Choose path f from x to y. Given loop α at x define $f_\#(\alpha) = \overline{f} \cdot \alpha \cdot f$ a loop at y. This well defined on homotopy classes:

$$\alpha \simeq_{\alpha_t} \beta \Rightarrow \overline{f} \cdot \alpha \cdot f \simeq_{\overline{f} \cdot \alpha_t \cdot f} \overline{f} \cdot \beta \cdot f$$

Gives a homomorphism $f_\#\colon \pi_1(X, x) \to \pi_1(X, y)$:

$$\begin{aligned} f_\#(\alpha \cdot \beta) &= \overline{f} \cdot \alpha \cdot \beta \cdot f \\ &\simeq \overline{f} \cdot \alpha \cdot f \cdot \overline{f} \cdot \beta \cdot f \\ &= f_\#(\alpha) \cdot f_\#(\beta). \end{aligned}$$

Claim $f_\#, \overline{f}_\#$ are inverse isomorphisms:

$$\begin{aligned}\overline{f}_\# \circ f_\#(\alpha) &= f \cdot (\overline{f} \cdot \alpha \cdot f) \cdot \overline{f} \\ &= (f \cdot \overline{f}) \cdot \alpha \cdot (f \cdot \overline{f}) \\ &\simeq \alpha.\end{aligned}$$

Hence $\overline{f}_\# \circ f_\# = Id_{\pi_1(X,x)}$. Similarly $f_\# \circ \overline{f}_\# = Id_{\pi_1(X,y)}$. □

5.3 Calculating $\pi_1(P^2)$

Lemma 5.14. *P^2 has path-lifting and homotopy lifting properties for $p\colon S^2 \to P^2$.*

Proposition 5.15.

$$\pi_1(P^2) \cong \mathbb{Z}_2.$$

Proof. Given loop $f\colon I \to P^2$ choose $\star \in P^2$ on equator then $p^{-1}(\star) = \{\star_1, -\star_1\}$. Consider $\tilde{f}$ lift starting at $\star_1$. Finishes at either $\star_1$ or $-\star_1$. Define

$$i(f) = \begin{cases} 0 & \tilde{f} \text{ finishes at } \star_1 \\ 1 & \tilde{f} \text{ finishes at } -\star_1. \end{cases}$$

i is homotopy invariant and if $i(f) = i(g) = 1$ then $i(f \cdot g) = 0$. Hence $i\colon \pi_1(P^2) \to \mathbb{Z}_2$ is homomorphism, clearly surjective. If $i(f) = 0$ then $\tilde{f}$ is a loop in S^2 but $\pi_1(S^2) = 0$ so $\tilde{f} \simeq c_{\star_1}$, so

$$f = p \circ \tilde{f} \simeq p \circ c_{\star_1} = c_\star$$

so i injective. Hence isomorphism. □

Definition 23. $f\colon (X,x) \to (Y,y)$, $g\colon (Y,y) \to (X,x)$ maps s.t. $g \circ f \simeq Id_{(X,x)}$, $f \circ g \simeq Id_{(Y,y)}$ then f, g are inverse *homotopy equivalences* and $(X,x), (Y,y)$ are *homotopy equivalent.*

Theorem 5.16. *If $f\colon (X,x) \to (Y,y)$, $g\colon (Y,y) \to (X,x)$ homotopy equivalences then f_*, g_* inverse isomorphisms between respective fundamental groups.*

Proof.

$$g_* \circ f_\star \underset{(5.4)}{=} (g \circ f)_* \underset{(5.6)}{=} \big(Id_{(X,x)}\big)_* \underset{(5.4)}{=} Id_{\pi_1(X,x)}.$$

Similarly $f_* \circ g_*$. □

Definition 24.

- $A \subset X$, map $r\colon X \to A$ s.t. $r|_A = Id_A$ is called a *retraction.*
- If $r \simeq_{r_t} Id_X$ through retractions (r_t a retraction $\forall\ t$) then r called a *deformation retraction.*

Proposition 5.17. *Deformation retraction is a homotopy equivalence.*

Proof. Let $i\colon A \hookrightarrow X$ inclusion. Then $r \circ i = Id_A$, $i \circ r \simeq_{r_t} Id_X$. If choose $\star \in A$ these are homotopy equivalences $(X, \star) \leftrightarrow (A, \star)$. ☐

Lemma 5.18. *There does not exist a retraction $D^2 \to S^1$.*

Proof. Suppose r retraction $D^2 \to S^1$, $i\colon S^1 \hookrightarrow D^2$. Then

$$\underbrace{\pi_1(S_1)}_{\cong \mathbb{Z}} \underset{i_*}{\longrightarrow} \underbrace{\pi_1(D_2)}_{=0} \underset{r_*}{\longrightarrow} \underbrace{\pi_1(S_1)}_{\cong \mathbb{Z}}$$

has $(r \circ i)_* = Id_* = Id_{\pi_1(S^1)}$ but

$$1 \underset{i_*}{\mapsto} 0 \underset{r_*}{\mapsto} 0.$$

Contradiction. ☐

Theorem 5.19 (Brouwer fixed point theorem)**.** *Any map $f\colon D^2 \to D^2$ has a fixed point.*

Proof. Suppose $\exists$ map with no fixed point. For $x \in D^2$ define $r(x) \in S^1$ by $f(x), x, r(x)$ lie on a straight line in that order. r is deformation retraction $D^2 \to S^1$. Contradiction by lemma. ☐

Theorem 5.20 (Fundamental group of a product)**.**

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y).$$

Proof. For $\alpha \in \pi_1(X \times Y)$ represent α by $f\colon I \to X \times Y$. Let

$$f_X = p_X \circ f, \quad f_Y = p_Y \circ f. \tag{5.i}$$

These loops in X, Y resp, and hence define $\alpha_1 \in \pi_1(X, x)$, $\alpha_2 \in \pi_1(Y, y)$. Conversely $\alpha_1 \in \pi_1(X, x)$, $\alpha_2 \in \pi_1(Y, y)$ together define $\alpha \in \pi_1(X \times Y, (x, y))$. This clearly well defined on homotopy classes so get bijection as required, homomorphism by (5.i). ☐

Corollary 5.21.

$$\pi_1(S^1 \times I) \cong \mathbb{Z}$$
$$\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2.$$

6 Covering Spaces

Assume all spaces are path connected in this section.

Definition 25. A map $p\colon E \to B$ is called a *covering space* if $\forall\, x \in B$ $\exists$ open $U \subset B$, $x \in U$ and a homeomorphism $p^{-1}(U) \to U \times F$ where F is a discrete space s.t. $p|\colon U \times f \to U$ is a homeomorphism for all $f \in F$.

Definition 26. If E is simply connected then $p\colon E \to B$ is called a *universal cover.*

Remark 1. If $p\colon E \to B$ a universal cover then $\pi_1(B)$" = "F.

Definition 27.
- A space is *0-connected* if it is path connected.
- A space is *1-connected* if it is simply connected.
- A space is *locally j-connected* ($j \in \{0,1\}$) if $\forall\, x \in X$, nhbd N of x $\exists$ smaller nhbd $N' \subset N$ s.t. N' is j-connected.

Example 3.
- $X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ is not locally 0-connected at 0.
- $X \times I \cup \{y = 1\} \subset \mathbb{R}^2$ is 0-connected but not locally 0-connected at 0.

Example 4 (Hawaiian earring)**.** $X = \left\{\text{circles radius } \frac{1}{n},\ \text{centre } \left(\frac{1}{n}, 0\right)\right\} \subset \mathbb{R}^2$ is locally 0-connected at 0 but not locally 1-connected as any nhbd of 0 contains complete circles, so has non-trivial π_1.

Proposition 6.1 (Path and Homotopy Lifting Properties)**.** *PLL and HLL ((5.8) and (5.9)) hold for any covering space $p\colon E \to B$ where B is locally 0-connected.*

Corollary 6.2. *$p_*\colon \pi_1(E) \to \pi_1(B)$ is injective.*

Proof. $\alpha\colon I \to E$ a loop, $p_*(\alpha) = e \in \pi_1(B)$. Then $p \circ \alpha \simeq_F c_{\star_B}$ so $\widetilde{p \circ \alpha} = \alpha \simeq_{\tilde{F}} c_{\star_E}$. □

Corollary 6.3. *Suppose E, B based, B locally 0-connected, $p\colon (E, \star_E) \to (B, \star_B)$ covering space and let $F = p^{-1}(\star_B)$. Then exists natural bijection $F \to \{$cosets of $\pi_1(E)$ in $\pi_1(B)\}$.*

Proof. For $[\alpha] \in \pi_1(B)$ lift α to $\tilde{\alpha}$ starting at $\star_E$. $\tilde{\alpha}$ ends at x, say in $p^{-1}(\star_B)$. By HLP x is independent of choice of $\alpha \in [\alpha]$.

This defines $\phi\colon \pi_1(B) \to F =: p^{-1}(\star_B)$. Is clearly onto. Suppose $[\beta] \in \pi_1(B)$ s.t. $\phi([\beta]) = x$ also. Let γ denote $\tilde{\beta}$ backwards. Then $\tilde{\alpha} \cdot \gamma$ is a loop in E, carried by p to $[\alpha][\beta]^{-1}$, so $[\alpha][\beta]^{-1} \in \pi_1(E)$. The converse is clear, so

$$\begin{aligned}\phi([\alpha]) = \phi([\beta]) &\iff [\alpha][\beta]^{-1} \in \pi_1(B) \\ &\iff [\alpha], [\beta] \text{ in same coset of } \pi_1(E) \text{ in } \pi_1(B).\end{aligned}$$ □

Theorem 6.4 (General Lifting Property)**.** *X locally 0-connected, $p\colon E \to B$ covering space, B locally 0-connected, $f\colon X \to B$ map, $x \in X$. Suppose $q \in E$, $p(q) = f(x)$. Then $\exists!\, \tilde{f}\colon X \to E$ s.t. $p \circ \tilde{f} = f$ and $\tilde{f}(x) = q$, provided*

$$f_*(\pi_1(X, x)) \subset p_*(\pi_1(E, q)) \tag{6.i}$$

where E, B based at $q, f(x)$ resp.

IDEA: Pick $y \in X$, path α from x to y. Lift $f \circ \alpha$ to path β starting at q, define $\tilde{f}(y) = \beta(1)$.

- Show $\tilde{f}$ well defined by choosing another path α' from x to y, show by lifting argument that $\alpha \cdot \overline{\alpha'}$ is a loop in X, then use (6.i) and (6.3) to show $\beta(1) = \beta'(1)$.

- Show $\tilde{f}$ cts by picking 0-connected nhbd of y, mapping into a chart of p. Join y' to y by a path α_1, then $\alpha \circ \alpha_1$ lifts to $\beta \circ \beta_1$ where β_1 path in a copy of U in $p^{-1}(U)$.

Corollary 6.5. *Suppose $p, p' \colon E, E' \to B$ both covering spaces, everything based, locally 0-connected, $\pi_1(E) \subset \pi_1(E')$ as subgroups of $\pi_1(B)$. Then $\exists! f \colon E \to E'$ map s.t. $f(\star_E) = \star_{E'}$ and $p' \circ f = p$.*

Corollary 6.6. *Suppose further that $\pi_1(E) = \pi_1(E')$ as subgroups of $\pi_1(B)$. Then $\exists\, g \colon E' \to E$, so by uniqueness $f \circ g = Id_{E'}$, $g \circ f = Id_E$.*

Theorem 6.7 (Classification thm)**.** *If B locally 1-connected there exists a bijection between equivalence classes of covers of B and subgroups of $\pi_1(B)$.*

Definition 28. $p \colon E \to B$ universal cover. A *covering transformation* of E is a homeomorphism $h \colon E \to E$ s.t. $p \circ h = p$.

7 Surfaces

Definition 29. A *surface* S is s.t. $\forall\, x \in S\ \exists$ open $U \subset S$, $x \in U$ and a homeomorphism $\phi \colon U \to U_1$ where $U_1 \subset \mathbb{R}^2$ open.

Will assume all our surfaces are *compact*, *metrizable*, *0-connected*.

Definition 30. S a surface. $D \subset S$ is a *disc* if $D \cong D^2$.

Definition 31. S_1, S_2 surfaces, $D_1 \subset S_1, D_2 \subset S_2$ discs, $h_1 \colon D^2 \to D_1$, $h_2 \colon D^2 \to D_2$ homeomorphisms. Define S_1 *connected sum* S_2,

$$S_1 \# S_2 = (S_1 \setminus h_1(D^\circ)) \cup_{h_2 \circ h_1^{-1}|_{\partial D}} (S_2 \setminus h_1(D^\circ))$$

where $D = D^2$.

Lemma 7.1 (Homogeneity)**.** *Given $x, y \in S$ exists homeomorphism $g \colon S \to S$ s.t. $g(x) = y$.*

Proof. For $x, y \in D^\circ$, $D^2 \cong D \subset S$ then $\exists$ homeomorphism given by cone construction of $D \to D$ carrying x to y, fixing ∂D. Extend by Id to S.

In general, join x to y by a path, cover by interiors of discs, use above repeatedly to carry x to y. □

Theorem 7.2 (Disc theorem)**.** *Let $h_1, h_2 \colon D^2 \to S$ be embedding. Then exists homeomorphism $g \colon S \to S$ s.t. $g \circ h_1 = h_2$.*

IDEA: Shrink $h_1(D^\circ)$ small, carry into $h_2(D^\circ)$ by lemma 7.1, expand to fit. This makes it fit setwise, pointwise comes from considering orientation.

Theorem 7.3 (Classification theorem)**.** *Any surface is homeomorphic to one of the following:*

S^2 $\quad T^2$ $\quad T^2 \# T^2$ $\quad \#_3 T^2$ $\quad \#_4 T^2$ $\quad \ldots$

P^2 $\quad K^2$ $\quad \#_3 P^2$ $\quad \#_4 P^2$ $\quad \#_5 P^2$ $\quad \#_6 P^2$ $\quad \ldots$

and no two on the list are homeomorphic.

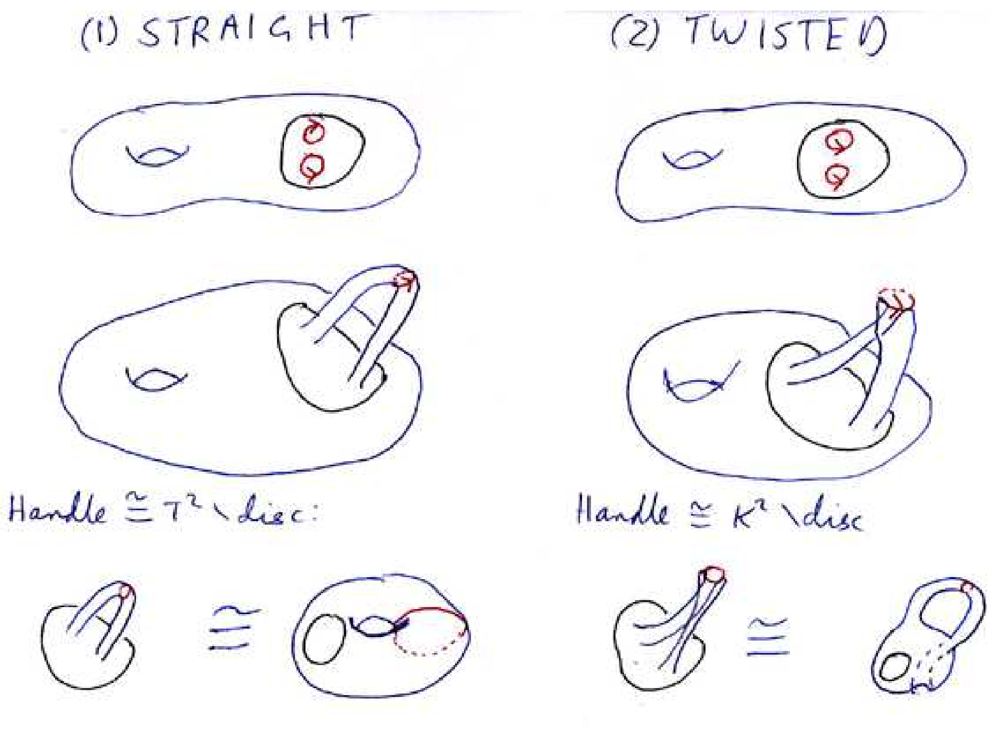

Figure 1: Adding a handle

Definition 32. *Adding a handle* means removing interiors of 2 disjoint discs in S, then gluing their boundaries together according to orientation.

Remark 2. In the straight case the handle is isomorphic to $T^2\backslash$disc and in the twisted case the handle is isomorphic to $K^2\backslash$disc so $S\cup$handle $= S\backslash$disc with either ($T^2\backslash$disc) or ($K^2\backslash$disc) glued in, i.e.

$$S^2\#T^2 \text{ or } S^2\#K^2.$$

Definition 33. A surface is *orientable* if there is a coherent sense of clockwise all over the surface.

Example 5.

- T^2 orientable
- M^2 non-orientable, since as you come around the turn "clockwise" flips.

Proposition 7.4. *A surface is non-orientable iff it contains a Möbius strip.*

Remark 3. If S non-orientable contains a Möbius strip so by above when adding a handle can transport a disc around a Möbius strip to replace a straight handle by a twisted one or v.v. Hence for non-orientable S, $S\#T^2 \cong S\#K^2$.

Example 6.

$$P^2\#T^2 \cong P^2\#K^2 = \#_3P^2.$$

Definition 34. Choose a non-separating simple closed curve $C \subset S$. Remove small nhbd of C. Then glue discs to the boundary components of resulting surface. Gives $Q(S,C)$, *surgery* on S along C.

Remark 4. **Case 1**

If a nhbd of C is an annulus get surface with 2 boundary components.

Case 2

If a nhbd of C is a Möbius strip get surface with 1 boundary component.

- In case 1, $S \cong Q(S,C)\#T^2$ or $Q(S,C)\#K^2$ (i.e. S obtained from $Q(S,C)$ by adding a handle).
- In case 2, $S \cong Q(S,C)\#P^2$ $\left(= \big(Q(s,C) \setminus (\text{interior of disc})\big) \cup_\partial M^2\right)$.

Definition 35. A *triangulated surface* is a collection of triangles in $\mathbb{R}^n$ s.t.

1. two triangles meet (if at all) in a common edge or a common vertex,
2. each edge is common to exactly 2 triangles and
3. each vertex lies in a simple circuit of triangles.

Definition 36. *Euler characteristic* χ of a triangulated surface is

$$V - E + F.$$

Definition 37.
- A triangle has two *orientations* - cyclic orderings of vertices.
- Orderings of adjacent triangles are *compatible* if they give opposite orientations for their common edge.
- A triangulated surface is *orientable* if it is possible to compatibly orient all triangles.

Theorem 7.5. *χ and orientability are topological invariants.*

Lemma 7.6.

$$\chi(S_1\#S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Proof. W.l.o.g. realise the connected sum using triangular regions and count:

- removing the triangles reduces $\chi(S_1) + \chi(S_2)$ by 2,
- gluing 3 vertices and 3 edges not change it. □

Corollary 7.7.

$$\chi(Q(S,C)) = \chi(S) + 1 \text{ or } + 2.$$

Proof. $S = Q(S,C)\#P^2$ or $\#K^2$ or $\#T^2$. (e.g. $\#T^2$ adds 10 edges, 8 vertices, 4 faces, so χ increases by $8 - 10 + 4 = 2$.) □

Theorem 7.8. *1. If S is triangulated surface then $\chi(S) \leq 2$.*

2. If $\chi(S) < 2$ then S contains a non-separating simple closed curve.

3. If $\chi(S) = 2$ then $S \cong S^2$.

Proof. Let T be a maximal tree in the edges of S and Γ its dual graph. Cut along Γ and push into plane. Γ does not separate S.

1.

$$\begin{aligned}\chi(S) &= V - E + F \\ &= V_T - (E_T + E_\Gamma) + V_\Gamma \\ &= (V_T - E_T) + (V_\Gamma - E_\Gamma) \quad = 1 + \alpha \quad (\alpha \leq 1 \text{ by lemma (1.1)}) \\ &\leq 2.\end{aligned}$$

2. Proof by contrapositive - Every simple closed curve in S separates $S \Rightarrow \chi(S) = 2$:

   Pick maximal tree T, dual Γ. Claim Γ a tree, so $\chi(S) = \chi(T) + \chi(\Gamma) = 2$. If not Γ contains loop so separates S. Each component of $S\backslash$loop contains vertex of T. Contradiction as T connected and disjoint from Γ. Hence Γ a tree.

3. $\chi(S) = 2$ then $\chi(\Gamma) = 1$ so Γ a tree. Then if cover S by nhbds N_T, N_Γ of T, Γ resp., these are discs D_1, D_2, say. Hence $S = D_1 \cup_\partial D_2 \cong S^2$. □

Proof of classification theorem (Theorem 7.3). If $\chi(S) < 2$ then by (2) can do surgery, which increases χ by corollary 7.7. Hence can only do surgery a finite number of times before you get S^2 by (3). Then can't do any more surgery or would get $\chi > 2$, contradicting (1). Running surgery backards you see S is # of S^2 and P^2, T^2, K^2, and $P^2\#T^2 \cong P^2\#K^2 = \#_3P^2$, so get

	S^2		T^2		$T^2\#T^2$		$\#_3T^2$	...	(orientable)
		P^2	K^2	$\#_3P^2$	$\#_4P^2$	$\#_5P^2$	$\#_6P^2$	...	(non-orientable)
$\chi =$	2	1	0	-1	-2	-3	-4	...	

and no two on the list are homeomorphic by results from Algebraic Topology. □