Spherical co-ordinates

http://en.wikipedia.org/wiki/Spherical_coordinates

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$$x(r,\varphi,\theta) = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta \\ r\sin\theta \end{pmatrix}$$

$r \in [0,\infty]$

$\varphi \in [0,2\pi)$

$\theta \in [\frac{-\pi}{2},\frac{\pi}{2}]$

$f(r,\varphi,\theta) = f(x(r,\varphi,\theta))$

we can define new basis vectors.

$$e_r = \frac{\delta x}{\delta r} = \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix}$$

$$||e_r|| = \sqrt{\cos^2\varphi\cos^2\theta + \sin^2\varphi\cos^2\theta + \sin^2\theta} = 1$$

$$e_\varphi = \frac{\delta x}{\delta\varphi} = \begin{pmatrix} -r\sin\varphi\cos\theta \\ r\cos\varphi\cos\theta \\ 0 \end{pmatrix}$$

$$|e_\varphi| = sqrt r^2\sin^2\varphi\cos^2\theta + r^2\cos^2\varphi\cos^2\theta = r|\cos\theta| = r\cos\theta$$

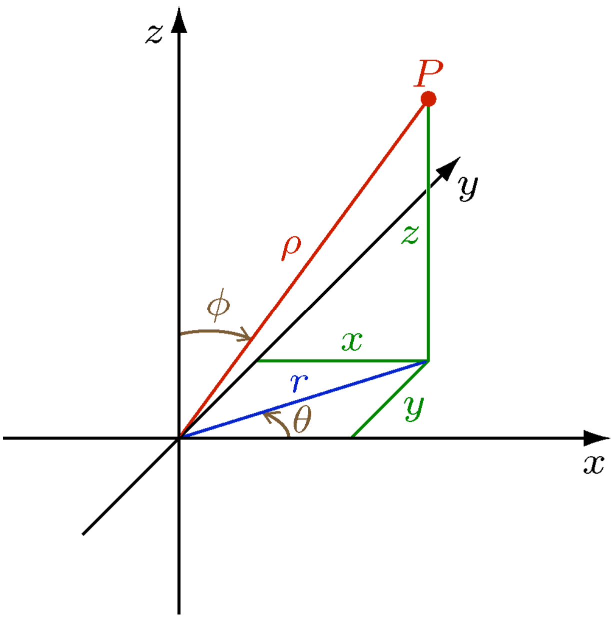
$$\hat{e}_\varphi = \frac{e_\varphi}{||e_\varphi||} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$e_\theta = \frac{\delta x}{\delta\theta} = \begin{pmatrix} -r\cos\varphi\sin\theta \\ -r\sin\varphi\sin\theta \\ r\cos\theta \end{pmatrix}$$

$$|e_\theta| = \sqrt{r^2\cos^2\varphi\sin^2\theta + r^2\sin^2\varphi\sin^2\theta + r\cos^2\theta} = r$$

$$\Rightarrow \hat{e}_\theta = \begin{pmatrix} -\cos\varphi\sin\theta \\ -\sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}$$

The system $(\hat{e}_r,\hat{e}_\varphi,\hat{e}_\theta)$ is a right handed orthogonal system.



Many vector fields can be naturally be expressed in this basis/

$v = v_r(r,\theta,\varphi)\hat{e}_r + v_\varphi(r,\theta,\varphi)\hat{e}_\varphi + v_\theta(r,\theta,\varphi)\hat{e}_\theta$

Example

$v(x) = x$

$v = r\hat{e}_r + 0\hat{e}_\varphi + 0\hat{e}_\theta$

Question: how to calculate divergence and gradient and curl in these co-ordinates?

(WARNING: $\mathrm{div} v \neq \frac{\delta v_r}{\delta r} + \frac{\delta v_\varphi}{\delta \varphi} + \frac{\delta v_\theta}{\delta \theta}$)

Solution 1

Find the inverse maps $(x,y,z) \mapsto r(x,y,z), \varphi(x,y,z), \theta(x,y,z)$ use the chain rule.

Solution 2 (use Gauss theorem)

Let $\Omega = [r, r+\Delta r] \times [\varphi, \varphi + \Delta\varphi] \times [\theta, \theta + \Delta\theta]$.

$\mathrm{div}(v) \approx \mathrm{div} v \cdot \Delta r ||e_r|| \cdot \Delta\varphi ||e_\varphi|| \cdot \Delta\theta ||e_\theta||$

$\approx \mathrm{div} v \cdot \mathrm{Vol}(\Omega)$

Flux through the front surface.

$$\int_\Omega \mathrm{div}(v) \approx \langle V, -e_\varphi \rangle \cdot \text{Area of front}$$

$= -V_\varphi(r,\varphi,\theta)\Delta r \Delta\theta$

back surface

$\approx V_\varphi(r, \varphi + \Delta\varphi, \theta)\Delta r \Delta\theta r$

Front + Back $(r(v(r, \varphi + \Delta\varphi, \theta)) - v(r, \varphi, \theta)) \cdot \Delta r \Delta\theta$

$\int_\varphi^{\varphi+\Delta\varphi} \frac{\delta v_\varphi}{\delta\varphi}(r, \tilde{\varphi}, \theta) d\tilde{\varphi} r \Delta r \Delta\theta$

$\approx \frac{\delta v_\varphi}{\delta\varphi}(r(\varphi, \theta)\Delta\varphi \cdot \Delta r \Delta\theta r)$

Similar for the left and right

$\approx \frac{\delta}{\delta r}(rV_r)\Delta r \cos\theta \Delta\varphi \Delta\theta$

Top +Bottom

$\approx \frac{\delta}{\delta\theta}(\cos\theta V_\theta)\Delta\theta \Delta r \Delta\varphi$

By Gauss's theorem

$\operatorname{div}(v) r^2 \cos\theta = \frac{\delta V_\varphi}{\delta\varphi} r + \frac{\delta}{\delta r}(rV_r)\cos\theta + \frac{\delta}{\delta\theta}(\cos\theta V_\theta)$